Matrix Exponentials

For a real number t we have

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

with the convention that when $t = 0$, the term $0^0 = 1$. Since we can think of real numbers as 1×1 real matrices, this gives us the idea to define e^A when A is an $N \times N$ matrix by the formula

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

with the convention that $A^0 = I$. The finite sums

$$\sum_{n=0}^{k} \frac{A^n}{n!}$$

all make sense for any integer $k \geq 0$ because we know how to add and multiply matrices and multiply them by constants. However, an infinite series is defined by a limit of finite sums, so we need to know what it means to take a limit of matrices.

By definition, we say that a limit of matrices exists if and only if the limit of each component exists, e.g.

$$\lim_{n\to\infty} M_n = M \iff \lim_{n\to\infty} M_n(i,j) = M(i,j) \quad \text{for all } i,j$$

or

$$\lim_{t\to t_0} M_t = M \iff \lim_{t\to t_0} M_t(i,j) = M(i,j) \quad \text{for all } i,j$$

and so on for all the types of limits we have defined. A more concrete example would be

$$\lim_{t\to 0} \begin{pmatrix} 1 & t \\ t^2+2 & e^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Thus our definition of e^A makes sense as long as

$$\lim_{k\to\infty} \sum_{n=0}^{k} \frac{A^n}{n!}$$

exists in the sense we just defined. We claim this is the case.

Theorem If A is an $N \times N$ real matrix, then for all i,j with $1 \leq i,j \leq N$

$$\sum_{n=0}^{\infty} \frac{A^n(i,j)}{n!} = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{A^n(i,j)}{n!}$$

is an absolutely convergent series of real numbers. In particular, e^A exists.

Lemma If A, B are $N \times N$ real matrices whose components are bounded in absolute value by a, b respectively, i.e. $|A(i,j)| \leq a, |B(i,j)| \leq b$ for all i,j, then all the components of AB are bounded by Nab.

Proof.

$$|AB(i,j)| = \left|\sum_{k=1}^{N} A(i,k)B(k,j)\right| \leq \sum_{k=1}^{N} |A(i,k)B(k,j)| \leq \sum_{k=1}^{N} ab = Nab. \quad \square$$

Proof. (Proof of Theorem.) Choose $a = \max_{i,j} |A(i,j)|$. Apply the lemma with A and A to find $|A^2(i,j)| \leq Na^2$ for all i,j. Apply the lemma again with A and A^2 to find that $|A^3(i,j)| \leq Na(Na^2) = N^2a^3$ for all i,j. Apply the lemma again with A and A^3 to find that $|A^4(i,j)| \leq Na(N^2a^3) = N^3a^4$ for all i,j. By repeating this argument (using induction) we find that for any power $n \geq 1$ we have $|A^n(i,j)| \leq N^{n-1}a^n$ for all i,j. Thus

$$\sum_{n=0}^{\infty} \frac{|A^n(i,j)|}{n!} = 1 + \sum_{n=1}^{\infty} \frac{|A^n(i,j)|}{n!} \leq 1 + \sum_{n=1}^{\infty} \frac{N^{n-1}a^n}{n!} \leq 1 + \sum_{n=1}^{\infty} \frac{N^n a^n}{n!} = e^{Na} < \infty$$

which shows the series is absolutely convergent. $\square$