

Exact Coupling of Random Walks on Polish Groups

JAMES T. MURPHY III*

The University of Texas at Austin

June 21, 2017

Abstract

Exact coupling of random walks is studied. Conditions for admitting a successful exact coupling are given that are necessary and in the Abelian case also sufficient. In the Abelian case, it is shown that a random walk S with step-length distribution μ started at 0 admits a successful exact coupling with a version S^x started at x iff there is n with $\mu^n \wedge \mu^n(x + \cdot) \neq 0$. In particular, this paper solves a problem posed by H. Thorisson on successful exact coupling of random walks on \mathbb{R} . It is also noted that the set of such x for which a successful exact coupling can be constructed is a Borel measurable group. Lastly, the weaker notion of possible exact coupling and its relationship to successful exact coupling is studied.

1. Introduction

Let G be a Polish group with identity e . If G is Abelian, additive notation is used instead and the identity is denoted 0. Let $S = \{S_n\}_{n=0}^\infty$ be a (right) random walk on G started at e and with *step-length distribution* μ , that is,

$$S_n = X_1 X_2 \cdots X_n, \quad 0 \leq n < \infty,$$

where the *step-lengths* $\{X_i\}_{i=1}^\infty$ are i.i.d. random elements in G with distribution μ . Let $S^x = \{S_n^x\}_{n=0}^\infty$ be a version of S started at $x \in G$, that is,

$$S_n^x = x X'_1 X'_2 \cdots X'_n, \quad 0 \leq n < \infty,$$

where $\{X'_i\}_{i=1}^\infty$ are i.i.d. with the same distribution of step-lengths μ . For shorthand, write that S and S^x are (e, μ) - and (x, μ) -*random walks on G* . An *exact coupling* of the random walks S and S^x defines them on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ in such a way that there is a random time T , called a *coupling time*, such that

$$S_n = S_n^x, \quad n \geq T.$$

*jamesmurphy@math.utexas.edu

If T is a.s. finite, the exact coupling is called *successful*. Language is abusively used as if S and S^x are already defined on some probability spaces and then redefined on a common space. Technically, new random walks \tilde{S} and \tilde{S}^x are defined on a common space, but it is linguistically simpler to use the same names S and S^x for the copies instead.

In this setting, admitting a successful exact coupling is equivalent to

$$\|\mathbf{P}(S_n \in \cdot) - \mathbf{P}(S_n^x \in \cdot)\|_{\text{TV}} \rightarrow 0, \quad n \rightarrow \infty,$$

where $\|\nu\|_{\text{TV}} = \sup_{F \in \mathcal{F}} |\nu(F)|$ denotes the total variation norm. In this setting, admitting a successful exact coupling is also equivalent to

$$\mathbf{P}(S \in E) = \mathbf{P}(S^x \in E), \quad E \in \mathcal{T},$$

where \mathcal{T} is the σ -algebra of tail measurable events. See Theorem 9.4 in [4] for the equivalence of these statements. In essence, successful exact coupling occurs iff the initial condition is uniformly forgotten as time progresses.

The results developed here are motivated by a question posed by Thorisson in [5]: for what initial positions x do the random walks admit a successful exact coupling? In [5], the question was posed only for $G := \mathbb{R}$, and in that case the following two special cases were known:

Theorem 1.1. [4] *Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on \mathbb{R} . The random walks admit a successful exact coupling for all $x \in \mathbb{R}$ iff the step-lengths are spread out, i.e. if there is $n \geq 1$ such that $\mathbf{P}(X_1 + \dots + X_n \in \cdot) \geq \int \cdot f d\lambda$ for some Borel $f \geq 0$ not Lebesgue-a.e. zero.*

Theorem 1.2. [1] *Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on \mathbb{R} , and suppose μ is purely atomic with A the set of atoms of μ . Then the random walks admit a successful exact coupling iff x is in the additive subgroup generated by $A - A$.*

For a Borel measure ν , denote the support of ν by $\text{supp } \nu$, the n -fold convolution of ν with itself (i.e. the law of an n -fold product of i.i.d. random variables with distribution ν if $\nu(G) = 1$) by ν^n , and the set of B for which $\nu(B) = 0$ by $\mathcal{N}(\nu)$. Given two measures ν_1 and ν_2 , denote $\nu_1 \wedge \nu_2$ to be the largest measure smaller than ν_1 and ν_2 . The zero measure is denoted 0. Note that for probability measures ν_1 and ν_2 , the conditions $\nu_1 \wedge \nu_2 \neq 0$, $\nu_1 \not\ll \nu_2$, $\|\nu_1 - \nu_2\|_{\text{TV}} < 1$, and the statement that the absolutely continuous part of ν_1 with respect to ν_2 is not the zero measure are all four equivalent. For $x \in G$, also define the shift $\theta_x \nu$ by $\theta_x \nu(B) := \nu(x^{-1}B)$. The interpretation of $\theta_x \nu$ is ν with all mass shifted (left-multiplied) by x , and $\theta_x \nu$ satisfies

$$\int_G f(y) \theta_x \nu(dy) = \int_G f(xy) \nu(dy)$$

for all Borel $f : G \rightarrow \mathbb{R}_+$.

The resolution to Thorisson's problem and generalizations of the previous theorems may now be stated. The proof is postponed and broken into several separate more general theorems appearing across multiple sections.

Theorem 1.3. *Suppose G is Abelian. Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on G respectively. Then the following hold:*

- (a) *S and S^x admit a successful exact coupling iff there is $n \geq 1$ such that $\mu^n \wedge \theta_x^{-1} \mu^n \neq 0$,*
- (b) *if G is connected and locally compact with Haar measure λ , and if μ is spread out, i.e. if there is $n \geq 1$ such that $\mu^n \geq \int f d\lambda$ for some Borel $f \geq 0$ not λ -a.e. zero, then S and S^x admit a successful exact coupling for all $x \in G$,*
- (c) *if μ is purely atomic with A the set of atoms of μ , then S and S^x admit a successful exact coupling iff x is in the subgroup generated by $A - A$.*
- (d) *the set of x for which S and S^x admit a successful exact coupling is a Borel measurable subgroup of G .*

2. Outline of the Paper

Section 3 builds to the main theorem, Theorem 3.5, which generalizes Theorem 1.3 (a). It is more technical but also applies in some non-Abelian cases. The reader familiar with the proof of the spread out case [4] or the purely atomic case [1] on \mathbb{R} may recognize the proof of Proposition 3.3, which shows that if μ^n dominates the sum of a measure ν and its shift $\theta_x^{-1} \nu$, then successful exact coupling can be achieved. Similarly, the proof of the main theorem of this document is similar to that of the purely atomic case on \mathbb{R} . The key insight allowing the proof of the main theorem is the reframing of when μ^n and $\theta_x^{-1} \mu^n$ overlap enough to fit a nontrivial measure underneath. The insight, given in Lemma 3.4, is elementary to prove but provides the final ingredient needed to connect partial results known in the spread out and purely atomic cases.

With the main theorem proved, Section 4 covers parts (a), (b), and (c) of Theorem 1.3 as simple corollaries. That is, it resolves the Abelian case and gives an even simpler description of the set of x for which successful exact coupling of S and S^x may occur in the spread out and purely atomic cases.

Section 5 then investigates the structure of the set of x for which S and S^x admit a successful exact coupling. In particular, it is easily seen to be a group. Through sufficient abstract nonsense, it is also shown to be Borel measurable. This shows part (d) of Theorem 1.3.

Finally, in Section 6, the weaker notion of possible exact coupling is studied. It is noted that the necessary and sufficient conditions derived for successful exact coupling in the Abelian case are coincidental. It is shown that the conditions derived for admitting a successful exact coupling in the Abelian case are, in the general case, equivalent to the ostensibly weaker notion of admitting a possible exact coupling, but that in the Abelian case admitting a possible exact coupling and admitting a successful exact coupling are equivalent. The paper ends with an example on a (non-Abelian) free group for which possible exact coupling can be done but successful exact coupling cannot.

3. The Main Theorem

This section culminates in the main theorem of the paper, Theorem 3.5, which gives necessary and sometimes sufficient conditions for successful exact coupling to occur, even in the non-Abelian case. The transfer and splitting theorems that appear in [4] are used. Less general versions are stated that are sufficient for the current setting. The need of a Polish space in the following is also the primary reason G is assumed to be Polish.

Theorem 3.1 (Transfer Theorem). [4] *Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and Y_1 is a random element in (E_1, \mathcal{E}_1) . Further suppose that there is a pair (Y'_1, Y'_2) on some probability space $(\Omega', \mathcal{F}', \mathbf{P}')$ with Y'_2 a random element in a Polish space (E_2, \mathcal{E}_2) , and Y_1 is a version of Y'_1 . Then Y'_2 can be transferred to $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. $(\Omega, \mathcal{F}, \mathbf{P})$ can be extended to accommodate a random element Y_2 which is conditionally independent of the original space given Y_1 , and with (Y_1, Y_2) having the same distribution as (Y'_1, Y'_2) . This transfer procedure can be repeated countably many times.*

Theorem 3.2 (Splitting Theorem). [4] *Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space and Y is a random element in (E, \mathcal{E}) . Let $\{\nu_i\}_{i=0}^\infty$ be subprobability measures on (E, \mathcal{E}) and suppose $\mathbf{P}(Y \in \cdot) \geq \sum_i \nu_i$. Then $(\Omega, \mathcal{F}, \mathbf{P})$ can be extended to accommodate a nonnegative integer-valued random variable K , called a splitting variable, such that $\mathbf{P}(Y \in \cdot, K = i) = \nu_i$. Moreover, K is conditionally independent of the original space given Y . This splitting operation can be repeated countably many times.*

The following is an extension of a result of Ö. Arnaldsson in [1] with nearly identical proof.

Proposition 3.3. *Let S and S^x be (e, μ) - and (x, μ) -random walks on G . Suppose that there is $n \geq 1$ such that $\mu^n \geq \nu + \theta_x^{-1}\nu$ for a nonzero measure ν . If G is Abelian or, more generally, if there is B with $B^c \in \mathcal{N}(\mu^n)$ such that x commutes with all of B , then the random walks admit a successful exact coupling with a coupling time T for which T/n has the same distribution as the hitting time of e of a lazy simple symmetric random walk on the cyclic group $\langle x \rangle$ started at x with probability $1 - 2\nu(G)$ of not moving at each step. In particular, $\mathbf{P}(T = n) = \nu(G)$.*

Proof. Begin with an $(\Omega, \mathcal{F}, \mathbf{P})$ housing S with step-lengths $\{X_i\}_{i=1}^\infty$. An extension of $(\Omega, \mathcal{F}, \mathbf{P})$ and an S^x on that extension are constructed such that successful exact coupling occurs. Let

$$L_i := X_{(i-1)n} \cdots X_{in}$$

for $i \geq 1$ so that $\{L_i\}_{i=1}^\infty$ is an i.i.d. family and $\mathbf{P}(L_i \in \cdot) \geq \nu + \theta_x^{-1}\nu$. By countably many applications of Theorem 3.2, expand $(\Omega, \mathcal{F}, \mathbf{P})$ to accommodate random variables $\{K_i\}_{i=1}^\infty$ taking values in $\{0, 1, 2\}$ such that $\{(L_i, K_i)\}_{i=1}^\infty$ is an i.i.d. sequence and

$$\mathbf{P}(L_i \in \cdot, K_i = 1) = \nu, \quad \mathbf{P}(L_i \in \cdot, K_i = 2) = \theta_x^{-1}\nu. \quad (1)$$

For $i \geq 1$ define

$$L'_i := \begin{cases} L_i, & K_i = 0, \\ x^{-1}L_i, & K_i = 1, \\ xL_i, & K_i = 2. \end{cases}$$

It is elementary to check using (1) that L'_i has the same distribution as L_i . Let R be the random walk started at e with step-lengths $\{L_i\}_{i=1}^\infty$, and let R' be the random walk started at x with step-lengths $\{L'_i\}_{i=1}^\infty$. By construction, $L'_i L_i^{-1} \in \{e, x, x^{-1}\}$. By assumption, it is possible to choose B with $\mu^n(B) = 1$ such that x commutes with all of B . Thus, a.s. every $L_i, L'_i \in B$ and so a.s. for every i ,

$$R'_i R_i^{-1} = x L'_1 \cdots L'_i L_i^{-1} \cdots L_1^{-1} = x (L'_1 L_1^{-1}) \cdots (L'_i L_i^{-1}) \in \langle x \rangle = \{x^m : m \in \mathbb{Z}\}.$$

Thus, $R' R^{-1}$ is in distribution the same as a lazy simple symmetric random walk started from x with step lengths $\{L'_i L_i^{-1}\}_{i=1}^\infty$. The walk has probability $\nu(G)$ to go left, $\nu(G)$ to go right, and $1 - 2\nu(G) < 1$ to stay put. Since nontrivial lazy simple symmetric random walks on cyclic groups are recurrent, there is an a.s. finite random time M with $R'_M R_M^{-1} = e$, i.e. the random walks R' and R meet at time M . Countably many applications of Theorem 3.1 make it possible to extend $(\Omega, \mathcal{F}, \mathbf{P})$ one final time to accommodate an i.i.d. sequence $\{X''_i\}_{i=1}^\infty$ with each X''_i having distribution μ and such that $L'_i = X''_{(i-1)n} \cdots X''_{in}$ for $i \geq 1$. Define $T := Mn$ and let S^x started at x with step-lengths

$$X'_i := \begin{cases} X''_i, & i \leq T, \\ X_i, & i > T. \end{cases}$$

Then S and S^x witness the definition of successful exact coupling with coupling time T . If $K_1 = 1$, then R and R' meet in one time step, so $T = n$, showing $\mathbf{P}(T = n) = \mathbf{P}(K_1 = 1) = \nu(G)$. \square

In the previous proof, the problem is reduced to the case where a difference process is a random walk. Since a general random walk may be transient, it is important to the proof that the difference process is made to be a random walk not on all of G , but rather on the cyclic group generated by x , so that the analysis reduces to that of \mathbb{Z} or $\mathbb{Z}/d\mathbb{Z}$. This highlights the fact that the joint distribution of S and S^x required to cause successful exact coupling is very special, and could not, except in trivial cases, be achieved with S and S^x being independent before the coupling time T .

The key insight allowing the main theorem to be proved follows. Its proof is elementary and may be skipped on first read.

Lemma 3.4. *For any probability measure ν on G ,*

$$\{x : \nu \wedge \theta_x^{-1} \nu \neq 0\} = \bigcap_{B^c \in \mathcal{N}(\nu)} BB^{-1}.$$

Proof. (\subseteq). Suppose x is such that $\nu \wedge \theta_x^{-1}\nu \neq 0$ and let B with $\nu(B^c) = 0$ be given. It cannot be that $\theta_x^{-1}\nu(B) = \nu(xB) = 0$, else $\nu \perp \theta_x^{-1}\nu$ which would imply $\nu \wedge \theta_x^{-1}\nu = 0$. Since $\nu(B) = 1$, it follows that $0 < \nu(xB) = \nu((xB) \cap B)$ so that $(xB) \cap B \neq \emptyset$. Writing $xb_1 = b_2$ for some $b_1, b_2 \in B$, it holds that $x = b_2b_1^{-1}$ and $x \in BB^{-1}$. Since B was arbitrary, $x \in \bigcap_{B^c \in \mathcal{N}(\nu)} BB^{-1}$.

(\supseteq). Suppose x is such that $\nu \wedge \theta_x^{-1}\nu = 0$. Then $\nu \perp \theta_x^{-1}\nu$, so choose A such that $\nu(A^c) = \nu(xA) = 0$. Then $\nu(A^c \cup (xA)) = 0$ as well, so choose $B := A \cap (xA)^c = (A^c \cup (xA))^c$. It must be that $x \notin BB^{-1}$. If not, write $x = b_2b_1^{-1}$ with $b_2, b_1 \in B$, from which it follows that $xb_1 = b_2$, showing that $b_2 \in xB \subseteq xA$, contradicting that $b_2 \notin xA$. Hence $x \notin BB^{-1}$. \square

The main theorem of the document follows. The first part of the proof is inspired by the proof of the purely atomic case in [1].

Theorem 3.5 (Main Theorem). *Let S and S^x be (e, μ) - and (x, μ) -random walks on G . If the random walks admit a successful exact coupling, then there is $n \geq 1$ such that $\mu^n \wedge \theta_x^{-1}\mu^n \neq 0$. Conversely, if $\mu^n \wedge \theta_x^{-1}\mu^n \neq 0$ for some $n \geq 1$ and there exists B with $B^c \in \mathcal{N}(\mu^n)$ such that x commutes with all of B , then the random walks admit a successful exact coupling with a coupling time T satisfying $\mathbf{P}(T = n) > 0$.*

Proof. Suppose that S and S^x witness the definition of successful exact coupling with coupling time T and respective step-lengths $\{X_i\}_{i=1}^\infty$ and $\{X'_i\}_{i=1}^\infty$. Choose n such that $\mathbf{P}(T = n) > 0$, which is possible since T is a.s. finite, and fix B with $B^c \in \mathcal{N}(\mu^n)$. With probability 1, $X_1X_2 \cdots X_n \in B$ and $X'_1X'_2 \cdots X'_n \in B$ so that on some $\omega \in \Omega$ for which $T(\omega) = n$, it holds that

$$x = (X_1X_2 \cdots X_n)(X'_1X'_2 \cdots X'_n)^{-1} \in BB^{-1}.$$

Since B was arbitrary, it follows that

$$x \in \bigcap_{B^c \in \mathcal{N}(\mu^n)} BB^{-1} = \{y : \mu^n \wedge \theta_y^{-1}\mu^n \neq 0\},$$

where the last equality follows from Lemma 3.4.

Conversely, suppose n is such that $\xi := \mu^n \wedge \theta_x^{-1}\mu^n \neq 0$. In case $x = e$, the random walks clearly admit a successful exact coupling with a coupling time $T := 0$, so assume $x \neq e$. Choose $y \in \text{supp } \xi$. Since $y \neq xy$, it is possible to choose a neighborhood U of y small enough that $U \cap xU = \emptyset$. Consider

$$\nu := \xi((x^{-1}\cdot) \cap U) \neq 0.$$

Then

$$\nu \leq \mu^n (x((x^{-1}\cdot) \cap U)) = \mu^n(\cdot \cap xU)$$

and

$$\theta_x^{-1}\nu = \xi(\cdot \cap U) \leq \mu^n(\cdot \cap U).$$

It follows that

$$\nu + \theta_x^{-1}\nu \leq \mu^n(\cdot \cap (U \cup xU)) \leq \mu^n.$$

Proposition 3.3 then shows the random walks admit a successful exact coupling with a coupling time T satisfying $\mathbf{P}(T = n) = \nu(G) > 0$. \square

4. The Abelian Case

In this section, parts (a), (b), and (c) of Theorem 1.3 are derived as simple corollaries of the main theorem. Firstly, the problem of determining for what x successful exact coupling of S and S^x occurs can be resolved entirely for Abelian G . This is part (a) of Theorem 1.3.

Corollary 4.1. *Suppose G is Abelian. Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on G . Then the random walks admit a successful exact coupling iff there is $n \geq 1$ such that $\mu^n \wedge \theta_x^{-1} \mu^n \neq 0$.*

Proof. Since G is Abelian, the condition in Theorem 3.5 that there is B with $B^c \in \mathcal{N}(\mu^n)$ such that x commutes with all of B is automatic. \square

Next, part (b) of Theorem 1.3 is covered. That is, when step-lengths are spread out, a successful exact coupling can always be achieved.

Corollary 4.2. *Suppose G is connected, locally compact, and Abelian with Haar measure λ . Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on G . The random walks admit a successful exact coupling for all $x \in G$ if μ is spread out, i.e. if there is $n \geq 1$ such that $\mu^n \geq \int f d\lambda$ for some Borel $f \geq 0$ not λ -a.e. zero.*

Proof. Suppose $\nu := \mu^n \geq \int f d\lambda$ as stated. By replacing f with $\min\{f, b\}1_K$ for some $b > 0$ and $K \subseteq G$ compact, one may assume f is bounded and compactly supported. Furthermore, it is claimed that by replacing n with $2n$ one may assume $f > \epsilon$ on some nonempty open set for some $\epsilon > 0$. Indeed,

$$\mu^{2n} = \nu * \nu \geq \int f * f d\lambda.$$

Since f is bounded and compactly supported, the convolution $f * f$ is continuous, and also $\|f * f\|_{L^1} = \|f\|_{L^1}^2 > 0$, so f is not constant 0. Thus the assumption that $f > \epsilon > 0$ on some nonempty open set U and for some $\epsilon > 0$ is justified. In particular, choosing a symmetric neighborhood V of the identity such that $(U - x) \cap U \neq \emptyset$ for each $x \in V$, it holds that

$$\nu \wedge \theta_x^{-1} \nu(G) \geq \int_G \min\{f(y), f(x + y)\} \lambda(dy) \geq \int_{(U-x) \cap U} \epsilon \lambda(dy) > 0$$

for every $x \in V$. It follows that the set H of x for which S and S^x admit a successful exact coupling contains V . As will be seen shortly in Corollary 5.2, H is a subgroup of G . Since H contains a nonempty open set V , H must contain the subgroup $\langle V \rangle$ generated by V , which is open, and hence clopen in G . The connectedness of G then implies $\langle V \rangle = G$, so that $H = G$ as well. \square

Whether or not the converse of Corollary 4.2 holds is not presently known in the setting of an arbitrary connected, locally compact, and Abelian group G . The converse is known to hold in the case $G := \mathbb{R}$.

The connectedness assumption in Corollary 4.2 plays a nontrivial role. For example, consider when G is a countable group. Then any choice of μ is automatically purely atomic because G is countable and spread out because λ is a counting measure. The following corollary shows that in that case the conclusion of Corollary 4.2 does not hold. This is also part (c) of Theorem 1.3.

Corollary 4.3. *Suppose G is Abelian and μ is purely atomic with A the set of atoms of μ . Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on G . Then the random walks admit a successful exact coupling iff x is in the subgroup $\langle A - A \rangle$ generated by $A - A$.*

Proof. The atoms of μ^n are $nA := A + \dots + A$. Then since μ^n is atomic, $\mu^n \wedge \theta_x^{-1} \mu^n \neq 0$ iff $nA \cap (nA - x) \neq \emptyset$ iff $x \in nA - nA = n(A - A)$. Finally, note that $\bigcup_{n=1}^{\infty} n(A - A)$ is exactly the subgroup generated by $A - A$ since $A - A$ is symmetric. Corollary 4.1 then finishes the claim. \square

5. The Successful Exact Coupling Set

In this section, the group structure and measurability of the set of x for which S and S^x admit a successful exact coupling is studied. The set of such x is henceforth called the *successful exact coupling set*. The following helps show that the successful exact coupling set is a group.

Proposition 5.1. *If $N \in \mathbb{N} \cup \{\infty\}$ and $\{S^{x_i}\}_{i=0}^N$ is a family with S^{x_i} an (x_i, μ) -random walk for each i such that each S^{x_i} admits successful exact coupling with $S := S^{x_0}$, then there exists a single probability space $(\Omega, \mathcal{F}, \mathbf{P})$ on which all S^{x_i} couple with S . That is, for every i there is an a.s. finite random time T_i with $S_n^{x_i} = S_n$ for all $n \geq T_i$.*

Proof. By assumption there is an successful exact coupling of S and S^{x_i} on some $(\Omega, \mathcal{F}, \mathbf{P})$. The extension procedure given by Theorem 3.1 can be repeated countably many times, once for each S^{x_i} , to give a single extension of $(\Omega, \mathcal{F}, \mathbf{P})$ on which S and S^{x_i} couple for every i . \square

Corollary 5.2. *Let S and S^x be (e, μ) - and (x, μ) -random walks. Let H denote the set of x for which S and S^x admit a successful exact coupling. Then H is a group.*

Proof. Consider $x, y \in H$. By Proposition 5.1, it is possible to define S, S^x, S^y on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ such that S, S^x witness the definition of successful exact coupling for x , and S, S^y witness the definition of successful exact coupling for y . Let T^x, T^y be the respective coupling times. Then for $n \geq \max\{T^x, T^y\}$ it holds that $S_n^y = S_n = S_n^x$. In particular, $x^{-1}S^x$ and $x^{-1}S^y$ witness the definition of successful exact coupling for $x^{-1}y$ with a coupling time $\max\{T^x, T^y\}$. Thus $x^{-1}y \in H$. \square

In the Abelian case, the successful exact coupling set is Borel measurable. To show this, some abstract nonsense, a slight but natural extension of Exercise 6.10.72 in [2], is required. The following gives the existence of a measurable choice of a family of Radon-Nikodym derivatives. Importantly, the following does not assume absolute continuity and instead produces Radon-Nikodym derivatives of the absolutely continuous parts of measures.

Proposition 5.3. *Let (X, \mathcal{A}, μ) be a finite measure space with \mathcal{A} countably generated, and let (T, \mathcal{B}) be a measurable space. Let $\{\mu_t\}_{t \in T}$ be any family of finite measures on X such that for each $A \in \mathcal{A}$, the function $t \mapsto \mu_t(A)$ is \mathcal{B} -measurable. Then there is an $\mathcal{A} \otimes \mathcal{B}$ -measurable $f : X \times T \rightarrow \mathbb{R}$ such that for every $t \in T$, $x \mapsto f(x, t)$ is a version of the Radon-Nikodym derivative of the absolutely continuous part of μ_t with respect to μ .*

Proof. First consider $X := [0, 1]$ and $\mathcal{A} := \mathcal{B}([0, 1])$, the Borel sets on $[0, 1]$. Fix a sequence $\{\epsilon_n\}_{n=0}^\infty$ with $\epsilon_n \searrow 0$. For every $t \in T$,

$$\lim_n \frac{\mu_t(B(x, \epsilon_n))}{\mu(B(x, \epsilon_n))} = \frac{d\mu_{t,a}}{d\mu}(x), \quad \mu\text{-a.e. } x, \quad (2)$$

where $\mu_{t,a}$ denotes the absolutely continuous part of μ_t with respect to μ . This follows from, e.g., Theorem 5.8.8. in [3]. Define

$$f(x, t) := \limsup_n \frac{\mu_t(B(x, \epsilon_n))}{\mu(B(x, \epsilon_n))}$$

for $x \in \text{supp } \mu$ and $t \in T$, and $f(x, t) := 0$ otherwise. By (2), it suffices to show f is $\mathcal{A} \otimes \mathcal{B}$ -measurable. Indeed, consider a fixed n and consider the numerator

$$(x, t) \mapsto \mu_t(B(x, \epsilon_n)) = \int_{[0,1]} 1_{|y-x| < \epsilon_n} \mu_t(dy).$$

Let $g(x, y) := 1_{|y-x| < \epsilon_n}$ and choose a sequence of measurable simple functions $\{s_k\}_{k=0}^\infty$ of the form

$$s_k(x, y) := \sum_{i=0}^{m_k} \alpha_{i,k} 1_{x \in A_{i,k}} 1_{y \in B_{i,k}},$$

with $0 \leq s_k \leq 1$ and $A_{i,k}, B_{i,k} \in \mathcal{B}([0, 1])$ for each k , and $s_k \rightarrow g$ as $k \rightarrow \infty$. Then

$$\int_{[0,1]} 1_{|y-x| < \epsilon_n} \mu_t(dy) = \lim_k \sum_{i=0}^{m_k} \alpha_{i,k} 1_{x \in A_{i,k}} \mu_t(B_{i,k}),$$

which shows $(x, t) \mapsto \mu_t(B(x, \epsilon_n))$ is a limit of $\mathcal{A} \otimes \mathcal{B}$ -measurable functions, showing its measurability. The argument for the denominator $(x, t) \mapsto \mu(B(x, \epsilon_n))$ is similar and easier. It follows that f is $\mathcal{A} \otimes \mathcal{B}$ -measurable.

Next, consider a general X and \mathcal{A} . Since \mathcal{A} is countably generated, choose an \mathcal{A} -measurable $\phi : X \rightarrow [0, 1]$ such that $\mathcal{A} = \{\phi^{-1}(B) : B \in \mathcal{B}([0, 1])\}$, cf. Theorem 6.5.5 in [2]. Also set

$$\nu := \mu(\phi \in \cdot), \quad \nu_t := \mu_t(\phi \in \cdot),$$

for each $t \in T$. For each $B \in \mathcal{B}([0, 1])$, it holds that $A := \phi^{-1}(B) \in \mathcal{A}$ and $t \mapsto \nu_t(B) = \mu_t(A)$ is \mathcal{B} -measurable. By the case where $X = [0, 1]$ and $\mathcal{A} = \mathcal{B}([0, 1])$, choose $f : [0, 1] \times T \rightarrow \mathbb{R}$ that is $\mathcal{B}([0, 1]) \otimes \mathcal{B}$ -measurable and such that for all $t \in T$, $f(\cdot, t)$ is a version of the Radon-Nikodym derivative of the absolutely continuous part of ν_t with respect to ν . Define $f_0 : X \times T \rightarrow \mathbb{R}$ by $f_0(x, t) := f(\phi(x), t)$. Then f_0 is $\mathcal{A} \otimes \mathcal{B}$ -measurable. Fix $t \in T$ and let $A \in \mathcal{A}$ be given. Choose $B \in \mathcal{B}([0, 1])$ with $A = \phi^{-1}(B)$. Then

$$\begin{aligned} \int_X 1_{x \in A} f_0(x, t) \mu(dx) &= \int_X 1_{\phi(x) \in B} f(\phi(x), t) \mu(dx) \\ &= \int_{[0, 1]} 1_{y \in B} f(y, t) \nu(dy) \\ &= \nu_{t, a}(B) \\ &= \mu_{t, a}(A). \end{aligned}$$

Some care should be taken in the last equality, where it is used that the absolutely continuous part of $\mu_t(\phi \in \cdot)$ with respect to $\mu(\phi \in \cdot)$ is the same as the push-forward with respect to ϕ of the absolutely continuous part of μ_t with respect to μ . Write

$$\nu_t = \nu_{t, a} + \nu_{t, s}, \quad \mu_t = \mu_{t, a} + \mu_{t, s},$$

with $\nu_{t, a} \ll \nu$ and $\nu_{t, s} \perp \nu$, and $\mu_{t, a} \ll \mu$ and $\mu_{t, s} \perp \mu$. Then also

$$\nu_t = \mu_t(\phi \in \cdot) = \mu_{t, a}(\phi \in \cdot) + \mu_{t, s}(\phi \in \cdot),$$

so it suffices to show by the uniqueness of Lebesgue decompositions that

$$\mu_{t, a}(\phi \in \cdot) \ll \nu \text{ and } \mu_{t, s}(\phi \in \cdot) \perp \nu.$$

Indeed, if $B \in \mathcal{B}([0, 1])$ is such that $0 = \nu(B) = \mu(\phi \in B)$, then $\mu_{t, a}(\phi \in \cdot) = 0$ because $\mu_{t, a} \ll \mu$. Thus $\mu_{t, a}(\phi \in \cdot) \ll \nu$. Similarly, choose $A \in \mathcal{A}$ such that $\mu_{t, s}(A^c) = \mu(A) = 0$. Choose $B \in \mathcal{B}([0, 1])$ with $A = \phi^{-1}(B)$, then compute $\mu_{t, s}(\phi \in B^c) = \mu_{t, s}(A^c) = 0$ and $\nu(B) = \mu(\phi \in B) = \mu(A) = 0$, so that $\mu_{t, s}(\phi \in \cdot) \perp \nu$. The previous use of $\nu_{t, a}(B) = \mu_{t, a}(A)$ is now justified, showing that $f_0(\cdot, t)$ is a version of the Radon-Nikodym derivative of the absolutely continuous part of μ_t with respect to μ , completing the claim. \square

Corollary 5.4. *For a probability measure ν on G , the set $\{x : \nu \wedge \theta_x^{-1} \nu \neq 0\}$ is Borel measurable. In particular, if S and S^x are $(0, \mu)$ - and (x, μ) -random walks on an Abelian G , then the successful exact coupling set $\bigcup_{n=1}^{\infty} \{x : \mu^n \wedge \theta_x^{-1} \mu^n \neq 0\}$ is Borel measurable.*

Proof. Apply Proposition 5.3 with $X := T := G$ and the family of measures $\nu_t := \theta_t^{-1} \nu$ for $t \in G$. For $A \subseteq G$ open and $t_n \rightarrow t \in G$, Fatou's lemma implies

that

$$\begin{aligned}
\nu_t(A) &= \int_G 1_{x \in tA} \nu(dx) \\
&= \int_G 1_{t^{-1} \in Ax^{-1}} \nu(dx) \\
&\leq \int_G \liminf_n 1_{t_n^{-1} \in Ax^{-1}} \nu(dx) \\
&\leq \liminf_n \int_G 1_{t_n^{-1} \in Ax^{-1}} \nu(dx) \\
&= \liminf_n \nu_{t_n}(A),
\end{aligned}$$

so that $t \mapsto \nu_t(A)$ is semicontinuous and hence measurable. A monotone class argument shows that $t \mapsto \nu_t(A)$ is measurable for all Borel $A \subseteq G$. Thus, Proposition 5.3 gives a measurable $f : G \times G \rightarrow \mathbb{R}$ such that for every $t \in G$, $x \mapsto f(x, t)$ is a version of the Radon-Nikodym derivative of the absolutely continuous part of $\theta_t^{-1}\nu$ with respect to ν . It follows that $M(t) := \int_G f(x, t) \nu(dx)$, the total mass of the aforementioned absolutely continuous part, is measurable. Hence

$$\{t : \nu \wedge \theta_t^{-1}\nu \neq 0\} = \{t : M(t) > 0\}$$

is measurable as well. □

6. Possible Exact Coupling

In this section, a weaker notion of coupling is studied. Suppose S and S^x are (e, μ) - and (x, μ) -random walks on G that admit an exact coupling with coupling time T . The coupling is a *possible exact coupling* if $\mathbf{P}(T < \infty) > 0$. The difference between possible exact coupling and successful exact coupling is that a possible exact coupling only requires $T < \infty$ with positive probability, whereas a successful exact coupling would require $T < \infty$ a.s. The *possible exact coupling set* refers to the set of x for which S and S^x admit a possible exact coupling.

Carefully looking over the proofs in Section 3 reveals that in many places, the fact that a coupling time T satisfies $T < \infty$ a.s. is used only to guarantee that $\mathbf{P}(T = n) > 0$ for some n , allowing the same proofs work for possible exact coupling as well. In particular, the following variations on Proposition 3.3 and Theorem 3.5 hold without the need for any kind of assumption about the existence of large sets that commute with x .

Proposition 6.1. *Let S and S^x be (e, μ) - and (x, μ) -random walks on G . Suppose that there is $n \geq 1$ such that $\mu^n \geq \nu + \theta_x^{-1}\nu$ for a nonzero measure ν . Then the random walks admit a possible exact coupling with a coupling time T satisfying $\mathbf{P}(T = n) = \nu(G)$.*

Proof. In the proof of Proposition 3.3, the only place where the assumption that there exists a B with $B^c \in \mathcal{N}(\mu^n)$ such that x commutes with all of B is needed is to show that the constructed coupling time T is a.s. finite and T/n looks like a hitting time of a random walk. When this assumption is not met, the coupling from that proof still works, and the coupling time T still satisfies $\mathbf{P}(T = n) = \nu(G)$, but not necessarily $\mathbf{P}(T < \infty) = 1$, and T/n does not necessarily look like a hitting time of a random walk on $\langle x \rangle$. \square

Theorem 6.2. *Let S and S^x be (e, μ) - and (x, μ) -random walks on G . The random walks admit a possible exact coupling with a coupling time T such that $\mathbf{P}(T = n) > 0$ iff $\mu^n \wedge \theta_x^{-1}\mu^n \neq 0$. In particular, the random walks admit a possible exact coupling iff there is an $n \geq 1$ such that $\mu^n \wedge \theta_x^{-1}\mu^n \neq 0$.*

Proof. The proof is nearly identical to that of Theorem 3.5, except one appeals to Proposition 6.1 to construct a possible exact coupling instead of Proposition 3.3. \square

One may now reap some low-hanging fruit. In particular, it is shown that the possible exact coupling set is Borel measurable, that in the Abelian case admitting a possible exact coupling and admitting a successful exact coupling are the same, and that if an n -fold convolution of a measure overlaps with one of its shifts, then all higher-fold convolutions of the measure admit the same property.

Corollary 6.3. *Let S and S^x be (e, μ) - and (x, μ) -random walks on G . Then the set of x for which S and S^x admit a possible exact coupling is Borel measurable.*

Proof. The set in question, by Theorem 6.2, equals $\bigcup_{n=1}^{\infty} \{y : \mu^n \wedge \theta_y^{-1}\mu^n \neq 0\}$, which is Borel measurable by Corollary 5.4. \square

Corollary 6.4. *Suppose G is Abelian. Let S and S^x be $(0, \mu)$ - and (x, μ) -random walks on G . Then S and S^x admit a possible exact coupling iff they admit a successful exact coupling.*

Proof. By Theorem 6.2 and Theorem 4.1 both are equivalent to $\mu^n \wedge \theta_x^{-1}\mu^n \neq 0$ for some $n \geq 1$. \square

Note that the previous corollary says that if an exact coupling with coupling time T satisfying $\mathbf{P}(T < \infty) > 0$ exists, then an exact coupling with coupling time T' with $\mathbf{P}(T' < \infty) = 1$ exists. It does not show that if $\mathbf{P}(T < \infty) > 0$ then $\mathbf{P}(T < \infty) = 1$.

Corollary 6.5. *For a probability measure ν on G , if $\nu^{n_0} \wedge \theta_x^{-1}\nu^{n_0} \neq 0$ for some $n_0 \geq 1$, then $\nu^n \wedge \theta_x^{-1}\nu^n \neq 0$ for all $n \geq n_0$.*

Proof. Let n_0 as above and let $n \geq n_0$ be given. By Theorem 6.2, the random walks S and S^x with step-length distribution ν started respectively at e and x admit a possible exact coupling with a coupling time T such that $\mathbf{P}(T = n_0) > 0$. Then $T' := T + (n - n_0)$ is also a coupling time for S and S^x with $\mathbf{P}(T' = n) > 0$, so by Theorem 6.2 it holds that $\nu^n \wedge \theta_x^{-1}\nu^n \neq 0$. \square

In the Abelian case, admitting a possible exact coupling and admitting a successful exact coupling turned out to be the same. Lastly, it is shown that in the non-Abelian case this is not necessarily the case.

Example 6.6. Let $G := \mathbb{F}_2$ be the free group on two letters a, b and consider S and S^{ab} simple random walks on G . That is, the step-length distribution μ is supported on four atoms:

$$\mu(\{a\}) = \mu(\{a^{-1}\}) = \mu(\{b\}) = \mu(\{b^{-1}\}) = \frac{1}{4}.$$

Suppose S starts at the empty word e , and S^{ab} starts at ab . If S and S^{ab} are taken to be independent, then with positive probability $S_1 = a = S_1^{ab}$, so a possible exact coupling can be easily constructed. Moreover, since the length of S is itself a Markov chain on \mathbb{Z} with positive bias, a.s. the length of S tends to ∞ and a limiting word is finalized. A similar statement holds for S^{ab} . Denote the limiting words $\lim S$ and $\lim S^{ab}$. Recall that admitting a successful exact coupling is equivalent to $\mathbf{P}(S \in \cdot) = \mathbf{P}(S^{ab} \in \cdot)$ on the tail σ -algebra \mathcal{T} . The set

$$\{s = \{s_n\}_{n=0}^{\infty} : \lim s \text{ starts with } b\}$$

is tail measurable. With τ the hitting time of e for S^{ab} , by the strong Markov property and the fact that at time τ it holds that S^{ab} starts anew as a copy of S ,

$$\begin{aligned} \mathbf{P}(\lim S^{ab} \text{ starts with } b) &= \mathbf{P}(\tau < \infty) \mathbf{P}(\lim S \text{ starts with } b) \\ &< \mathbf{P}(\lim S \text{ starts with } b). \end{aligned}$$

Thus there is no successful exact coupling between S and S^{ab} .

7. Acknowledgments

This work was supported by a grant of the Simons Foundation (#197982 to The University of Texas at Austin).

References

- [1] Arnaldsson, Ö.: On coupling of discrete random walks on the line (2010)
- [2] Bogachev, V.I.: Measure theory, vol. 2. Springer Science & Business Media (2007)
- [3] Bogachev, V.I.: Measure theory, vol. 1. Springer Science & Business Media (2007)
- [4] Thorisson, H.: Coupling, stationarity, and regeneration, vol. 200. Springer New York (2000)
- [5] Thorisson, H.: Open problems in renewal, coupling and palm theory. *Queueing Systems* **68**(3-4), 313–319 (2011)