EXISTENCE FOR THE DIRICHLET PROBLEM IN ZAREMBA DOMAINS

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1. Introduction and Review

This text was written to be given as a half-hour talk. The majority of the material was amalgamated, if not directly taken, from [3]. In this paper $D \subseteq \mathbb{R}^d$ is nonempty, open, and bounded.

Definition 1.1. *We say D is a* Zaremba domain *if for all $a \in \partial D$, a locally satisfies* Zaremba's cone condition*, i.e. there exists a nonzero $y \in \mathbb{R}^d$, $\theta \in (0, \pi)$, and $r > 0$ such that inside $B(a, r)$ the translated cone $a + C(y, \theta)$ in the direction of y of aperture θ is contained in $\mathbb{R}^d \setminus D$. Explicitly, $C(y, \theta) := \left\{x \in \mathbb{R}^d : x \cdot y \geqslant \|x\|\|y\| \cos\theta\right\}$, and we require $(a + C(y, \theta)) \cap B(a, r) \subseteq \mathbb{R}^d \setminus D$.*

Definition 1.2. *Given $f \in C(\partial D)$, we say that $u \in C(\overline{D}) \cap C^2(D)$ solves the Dirichlet problem in D with boundary data f if*

$$\begin{cases} \triangle u(x) = 0 & x \in D \\ u(x) = f(x) & x \in \partial D \end{cases} \tag{1}$$

Our aim in this paper is to prove the following.

Theorem 1.3. *If D is a Zeremba domain and $f \in C(\partial D)$, then there exists a solution to the Dirichlet problem in D with boundary data f.*

We will go about this using stochastic methods to prove there is a locally integrable u with the mean value property that approaches f near the boundary.

Definition 1.4. *A locally integrable function $u : D \to \mathbb{R}$ is said to have the* mean value property *if for all balls $B(x, r) \subseteq D$, we have*

$$u(x) = \frac{1}{|\partial B(x, r)|} \int_{\partial B(x,r)} u(y)\, dS(y).$$

It will then follow that u is harmonic in D and therefore the unique solution to the Dirichlet problem (1). See either of [2, 1] for a more in-depth study of harmonic functions and their connection to the mean value property, and see either of [3, 4] for a more in-depth treatment of Brownian motion.

2. Proof of the Main Theorem

We begin by introducing the necessary probabilistic framework. Once and for all, fix $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t \in \mathbb{R}^+}, P, \{P^x\}_{x \in \mathbb{R}^d}, W)$ the canonical d-dimensional Brownian family on Weiner space under the universal filtration generated by W. In particular, for all $x \in \mathbb{R}^d$ the coordinate process W is a d-dimensional Brownian motion starting at x under P^x. Since we work on the canonical space we have the shift operators $\theta_t : \Omega \to \Omega$ defined by $W_s(\theta_t(\omega)) = \theta_t(\omega)(s) := \omega(s+t) = W_{s+t}(\omega)$. Moreover, the Brownian family is a strong Markov family. In particular, for any $x \in \mathbb{R}^d$, any Y that is $\mathscr{F}_\infty$ measurable and bounded, and any a.s. finite $\{\mathscr{F}_t\}$- optional time S, we have $E^x[Y \circ \theta_S | \mathscr{F}_{S+}] = E^{W_S}[Y]$ a.s. under P^x.

Let $\tau_D := \inf\left\{t \geqslant 0 : W_t \in \mathbb{R}^d \setminus D\right\}$ denote the exit time of W from D. Since D is bounded, for all $x \in \mathbb{R}^d$, we know $E^x[\tau_D] < \infty$. Thus we are in a position to propose the ansatz

$$u(x) := E^x[f(W_{\tau_D})], \quad x \in \overline{D}. \tag{2}$$

Step 2.1. *u has the mean value property, and is thus C^∞ on D.*

Proof. It is immediate since f is bounded that u is bounded, and hence locally integrable. Let $B(x,r) \subseteq D$. Then the exit time $\tau_{B(x,r)}$ from $B(x,r)$ satisfies $\tau_{B(x,r)} \leqslant \tau_D$, and we compute

$$\begin{aligned} u(x) &= E^x[f(W_{\tau_D})] \\ &= E^x[E^x[f(W_{\tau_D})|\mathscr{F}_{\tau_{B(x,r)}+}]]. \end{aligned}$$

Then we note that by properties of the shift operators

$$\begin{aligned} f(W_{\tau_D}) \circ \theta_{\tau_{B(x,r)}} &= f(W_{\tau_{B(x,r)}+\tau_D\circ\theta_{\tau_{B(x,r)}}}) \\ &= f(W_{\tau_D}) \end{aligned}$$

so that

$$\begin{aligned} u(x) &= E^x[E^x[f(W_{\tau_D}) \circ \theta_{\tau_{B(x,r)}}|\mathscr{F}_{\tau_{B(x,r)}+}]] \\ &= E^x[E^{W_{\tau_{B(x,r)}}}[f(W_{\tau_D})]] \\ &= E^x[u(W_{\tau_{B(x,r)}})] \\ &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dS(y) \end{aligned}$$

where here we used the strong Markov property, as well as the rotational invariance of Brownian motion. □

Step 2.2. *Let $\sigma_D := \inf\left\{t > 0 : W_t \in \mathbb{R}^d \setminus D\right\}$. Then all $a \in \partial D$ are* regular*, i.e. we have $P^a[\sigma_D = 0] = 1$. This is the crucial step where we use the fact that D is a Zaremba domain, which intuitively implies that if started on ∂D, W isn't going to immediately enter D and stay there.*

Proof. Fix $a \in \partial D$. Note that regularity at a is a local condition. If a is regular for the domain $D \cap B(a,r)$ for some $r > 0$, then it is regular for D. This is immediate because if a is regular for $D \cap B(a,r)$, then we have the P^a-a.s. pathwise statement that W hits points outside of $D \cap B(a,r)$ at arbitrarily small times $t > 0$. But W has continuous paths, so for a fixed path W stays inside $B(a,r/2)$ for all small times, and hence it must be that actually W is exiting D at arbitrarily small times $t > 0$. We may therefore assume by replacing D with some $D \cap B(a,r)$ if necessary that D satisfies the Zaremba cone condition without the local stipulation, i.e. we may choose $y \neq 0$ and $\theta \in (0,1)$ such that $a + C(y,\theta) \subseteq D$.

Then we notice that the cone $C(y,\theta)$ is invariant under dilation by any factor $t > 0$. Thus for any $t > 0$

$$\begin{aligned} P^a[\sigma_D \leqslant t] &\geqslant P^a[W_t \in a + C(y,\theta)] \\ &= P^0[W_t \in C(y,\theta)] \\ &= \int_{C(y,\theta)} \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{\|x\|^2}{2t}\right) dx \\ &= \int_{C(y,\theta)} \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{\|z\|^2}{2}\right) dz \\ &=: p \end{aligned}$$

where we used the substitution $z = x/\sqrt{t}$. Thus $P^a[\sigma_D \leqslant t] \geqslant p > 0$ for all $t > 0$. Sending $t \to 0$ we find $P^a[\sigma_D = 0] > 0$. But $\{\sigma_D = 0\} \in \mathscr{F}_{0+}$, so the Blumenthal zero-one law implies $P^a[\sigma_D = 0] = 1$. □

Step 2.3. *For all $a \in \partial D$ and $\epsilon > 0$ we have*

$$\lim_{\substack{x \to a \\ x \in D}} P^x[\tau_D > \epsilon] = 0.$$

Proof. Fix $a \in \partial D$ and $\epsilon > 0$. Since $\tau_D \leqslant \sigma_D$ we have

$$\limsup_{\substack{x \to a \\ x \in D}} P^x[\tau_D > \epsilon] \leqslant \limsup_{\substack{x \to a \\ x \in D}} P^x[\sigma_D > \epsilon].$$

We claim that $x \mapsto P^x[\sigma_D > \epsilon]$ is upper semi-continuous. If that is the case, we would then have

$$\begin{aligned}\limsup_{\substack{x \to a \\ x \in D}} P^x[\tau_D > \epsilon] &\leqslant P^a[\sigma_D > \epsilon] \\ &\leqslant P^a[\sigma_D \neq 0] \\ &= 0\end{aligned}$$

where the last equality is by the previous step 2.2. Thus it suffices to show that $x \mapsto P^x[\sigma_D > \epsilon]$ is upper semi-continuous. The difficulty lies in the fact that the distribution of W_0 is Dirac. We give W some room to diffuse; fix $\delta \in (0, \epsilon)$. By the strong Markov property[1]

$$\begin{aligned}P^x[\forall s \in [\delta, \epsilon], W_s \in D] &= E^x[E^x[1_{\forall s \in [\delta,\epsilon], W_s \in D} | \mathscr{F}_{\delta+}]] \\ &= E^x[E^x[1_{\forall t \in [0,\epsilon-\delta], W_t \in D} \circ \theta_\delta | \mathscr{F}_{\delta+}]] \\ &= E^x[E^{W_\delta}[1_{\forall t \in [0,\epsilon-\delta], W_t \in D}]].\end{aligned}$$

Since $\delta > 0$, W_δ now has a very smooth distribution. Continuing the calculation

$$P^x[\forall s \in [\delta, \epsilon], W_s \in D] = \int_{\mathbb{R}^d} E^y[1_{\forall t \in [0,\epsilon-\delta], W_t \in D}] \frac{1}{(2\pi\delta)^{d/2}} \exp\left(\frac{\|y - x\|^2}{2\delta}\right) dy$$

and so we see that $x \mapsto P^x[\forall s \in [\delta, \epsilon], W_s \in D]$ is in fact continuous on $\mathbb{R}^d$. Sending $\delta \to 0^+$ we find

$$x \mapsto P^x[\forall s \in (0, \epsilon], W_s \in D] = P^x[\sigma_D > \epsilon]$$

is the infimum of a family of continuous functions, and is thus upper semi-continuous. □

Step 2.4. *For all $a \in \partial D$ we have*

$$\lim_{\substack{x \to a \\ x \in D}} u(x) = f(a).$$

Proof. Let m be an increasing modulus of continuity for f on ∂D. Then for any $r > 0$

$$\begin{aligned}|u(x) - f(a)| &= |E^x[f(W_{\tau_D}) - f(a)]| \\ &\leqslant E^x[|f(W_{\tau_D}) - f(a)| 1_{|W_{\tau_D} - a| < r}] + E^x[|f(W_{\tau_D}) - f(a)| 1_{|W_{\tau_D} - a| \geqslant r}] \\ &\leqslant m(r) P^x[|W_{\tau_D} - a| < r] + 2\|f\|_\infty P^x[|W_{\tau_D} - a| \geqslant r].\end{aligned}$$

Taking the limsup as $x \to a$ with $x \in D$, then sending $r \to 0^+$ shows that it suffices to prove

$$\lim_{\substack{x \to a \\ x \in D}} P^x[|W_{\tau_D} - a| < r] = 1 \tag{3}$$

[1] With $D_r := \{x \in D : d(x, \mathbb{R}^d \setminus D) > r\}$, D_r is open and hence $\{\forall s \in [\delta, \epsilon], W_s \in D\} = \bigcup_{q, r \in \mathbb{Q} \cap (0,1)} \{\forall s \in [\delta, \epsilon] \cap \mathbb{Q}, W_s \in D_r\}$ is measurable.

for all $r > 0$. Fix $r > 0$ and $\epsilon > 0$. Then for any $x \in D$ if $|x - a| < r/2$ we have

$$\begin{aligned} P^x[|W_{\tau_D} - a| < r] &\geqslant P^x[|W_{\tau_D} - x| + |x - a| < r] \\ &\geqslant P^x[|W_{\tau_D} - x| < r/2] \\ &\geqslant P^x[\tau_D \leqslant \epsilon, \max_{t\in[0,\epsilon]} |W_t - x| < r/2] \\ &= P^x[\max_{t\in[0,\epsilon]} |W_t - x| < r/2] - P^x[\tau_D > \epsilon, \max_{t\in[0,\epsilon]} |W_t - x| < r/2] \\ &\geqslant P^x[\max_{t\in[0,\epsilon]} |W_t - x| < r/2] - P^x[\tau_D > \epsilon] \\ &= P^0[\max_{t\in[0,\epsilon]} |W_t| < r/2] - P^x[\tau_D > \epsilon]. \end{aligned}$$

Taking the liminf as $x \to a$ with $x \in D$ and using step 2.3, then sending $\epsilon \to 0^+$ gives (3), completing the proof of this step, and of the main theorem 1.3. □

3. References

[1] L. Evans, *Partial differential equations: Graduate studies in mathematics*, American Mathematical Society, 2 (1998).

[2] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, vol. 224, springer, 2001.

[3] I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Springer, 2 ed., 1991.

[4] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, vol. 293, Springer Science & Business Media, 1999.