Resolving the one-dimensional autonomous flow-free explosion problem

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Abstract

Our main result is the explicit calculation of the explosion threshold λ^* for the one-dimensional flow-free case $-\phi'' = \lambda g(\phi)$. More generally, we consider the d-dimensional explosion problem in a flow, and we give explicit bounds on $\lambda^*(u)$ for not necessarily divergence-free flows u. We include a PDE proof of a well-known fast-flow convergence result in two dimensions that states that stirring quickly causes solutions to converge uniformly to a function that is constant on stirring streamlines. We also give several example applications of stochastic analysis and probability to other areas of mathematics. Notably, we give short probabilistic proofs of versions of the weak and strong maximum principles from PDEs, Liouville's theorem from complex analysis, and Euler's product formula from number theory. Finally, we end with a brief overview of the theory of viscosity solutions for second order PDEs.

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1 Introduction

1.1 The explosion problem

In its most general form, the explosion problem is a question about positive solutions to semilinear elliptic PDEs of the form

$$\begin{cases} -L\phi = \lambda g(x,\phi) & x \in \Omega \\ \phi(x) = f(x) & x \in \partial\Omega \end{cases} \tag{1.1}$$

and parabolic PDEs, possibly with different types of boundary conditions, of the form

$$\begin{cases} \partial_t \phi - L\phi = \lambda g(x,t,\phi) & (x,t) \in \Omega \times (0,T) \\ \phi(x,0) = f(x) & (x,0) \in \Omega \times \{0\} \\ \phi(x,t) = b(x,t) & (x,t) \in \partial\Omega \times (0,T). \end{cases} \tag{1.2}$$

We would like to understand under what conditions these PDEs have solutions, have unique, maximal, or minimal solutions, what kind of regularity solutions have, and fast numerical schemes to compute solutions when they exist.

An interesting observation is that PDEs of the form (1.1) or (1.2) often exhibit dichotomous behavior depending on whether the positive parameter λ is large or small. In particular, under certain conditions, there exists a critical $\lambda^* > 0$, called the *explosion threshold*, such that

(a) in the elliptic case, for all $\lambda < \lambda^*$ the problem (1.1) has at least one nonnegative solution, while for all $\lambda > \lambda^*$ there are no nonnegative solutions to (1.1), and

(b) in the parabolic case, for all $\lambda < \lambda^*$ the problem (1.2) has at least one nonnegative solution which is global in time ($T = \infty$), while for all $\lambda > \lambda^*$, any solution to (1.2) blows up in finite time.

We will be precise about the exact assumptions we make for each problem that we consider. However, we provide generic versions of these assumptions which are commonly seen. Frequently, Ω is taken to be a bounded, open, simply connected domain in $\mathbb{R}^d$ with smooth boundary. Positivity, monotonicity, and convexity assumptions are made on g, as well as fast growth conditions. For the case $g = g(\phi)$, fast growth usually means one of $g'(s) \to \infty$, $g(s)/s \to \infty$, or $\int_0^\infty \frac{1}{g(s)}\,ds < \infty$, and the canonical examples to keep in mind are $g(s) := e^s$ and $g(s) := (1+s)^p$ for $p > 1$. The linear operator L is typically taken uniformly elliptic of the form $L := \triangle - u \cdot \nabla$ for a smooth divergence-free vector field u which is tangential to $\partial\Omega$. In this situation, one may view the elliptic and parabolic problems respectively as steady-state and time-dependent advection-diffusion systems in an incompressible fluid with nonlinear source terms.

The existence of the critical λ^* for the elliptic case was largely developed by Keener and Keller [23], Joseph and Lundgren [19], and Crandall and Rabinowitz [7]. As for the parabolic case, Sattinger showed in [31] the relationship between lower and upper solutions of the elliptic problem and global solutions to the parabolic problem. Namely, if $\underline{\phi}$ and $\overline{\phi}$ are lower and upper solutions to the elliptic problem respectively, then for any continuous initial data φ with $\underline{\phi} \le \varphi \le \overline{\phi}$, there is a global solution to the parabolic problem with initial data φ which stays bounded between $\underline{\phi}$ and $\overline{\phi}$. Moreover, [31] also showed that with zero boundary conditions, if there is a global solution ϕ to the parabolic problem with initial data $\varphi := \underline{\phi}$, where $\underline{\phi}$ is a lower solution to the elliptic problem, then $\phi \nearrow \tilde{\phi}$ as $t \to \infty$, where $\tilde{\phi}$ is a solution to the elliptic problem. Note that since g is assumed to be positive, $\underline{\phi} := 0$ will always be a lower solution to the elliptic problem, so the previous statement can at least be applied in this case.

More recently, effort has been spent to understand how mixing with different incompressible flows affects the explosion threshold. Intuitively, the explosion threshold exists because hotspots develop in the fluid. When λ is large, these hotspots become so hot that their existence would imply that the solution to the elliptic problem must be $\phi := +\infty$ everywhere. This also gives reason to why we focus mainly on divergence-free flows. If we were allowed to compress our fluid towards a point, surely we could make a hotspot as hot as we want. Amongst incompressible flows however, compression towards a point is not possible since divergence-free flows preserve volume.

Furthermore, stirring by an incompressible flow u is actually known to improve diffusivity in the sense that the principal eigenvalue $\mu_1[u]$ of the problem

$$\begin{cases} -\triangle \phi^u + u \cdot \nabla \phi^u = \mu_1[u]\phi^u & x \in \Omega \\ \phi^u(x) = 0 & x \in \partial\Omega \end{cases}$$

is never larger than the corresponding $\mu_1[0]$ for $u := 0$. From this intuition we may expect that the explosion threshold $\lambda^*(u)$ with stirring by an incompressible u is always larger than the explosion threshold $\lambda^*(0)$ with no stirring. Novikov showed in [27] that when Ω is a ball, indeed $\lambda^*(0) \leq \lambda^*(u)$ for incompressible flows u. Surprisingly, this does not appear to be the case when Ω is not a ball. Kagan et. al. showed numerically in [20] that in a long thin rectangle with $g(s) := e^s$ there is an incompressible u with $\lambda^*(u) < \lambda^*(0)$. Furthermore, Iyer et. al. showed in [18] that with $g(s) := 1$, unless Ω is a ball, there is an incompressible u which creates hotspots in the sense that $\|\phi^u\|_\infty > \|\phi^0\|_\infty$. In [4], Berestycki et. al. studied how much stirring can change the explosion threshold. In particular, they presented bounds on $\lambda^*(u)$ that are independent of the advecting flow u, and they characterized when the explosion threshold tends to ∞ under fast stirring.

2 The one-dimensional flow-free case $-\phi'' = \lambda g(\phi)$

2.1 A formula for λ^*

We direct our attention to the one-dimensional case. Say $\Omega := (a, b)$ with $a < b$. In one dimension, the only divergence-free u are constants, and of these only $u := 0$ is tangential to $\partial\Omega$. Hence we will consider the explosion problem for

$$\begin{cases} -\phi'' = \lambda g(\phi) & x \in (a,b) \\ \phi(x) = 0 & x \in \{a, b\}\,. \end{cases} \tag{2.1}$$

We claim that we can go so far as to provide the form of any nonnegative solution ϕ of (2.1), if one exists, for any non-constant, continuous, increasing, convex $g : [0, \infty) \to (0, \infty)$, and that this will allow us to explicitly compute λ^*. First we need a regularity lemma about solutions to (2.1), then we state the main theorem.

Lemma 2.1. *Any solution ϕ of (2.1) satisfies $\phi \in C^2([a, b])$ so that $-\phi'' = \lambda g(\phi)$ holds on $[a, b]$. Moreover, if $g \in C^k([0, \infty))$ for $k \geq 0$, then $\phi \in C^{k+2}([a, b])$. The only assumption on g we require for this lemma is continuity.*

Theorem 2.2. *There is $\lambda^* > 0$ such that for all $\lambda < \lambda^*$ the problem (2.1) has a minimal nonnegative solution, and for all $\lambda > \lambda^*$ the problem (2.1) has no nonnegative solutions. Furthermore, λ^* is given by*

$$\lambda^* = \sup_{k>0} \frac{2}{(b-a)^2} H_k(G^{-1}(k))^2 \tag{2.2}$$

where

$$G(u) := \int_0^u g(s)\, ds, \quad H_k(y) := \int_0^y \frac{du}{\sqrt{k - G(u)}},$$

and when $\lambda \leq \lambda^*$, ϕ solves (2.1) if and only if ϕ is given by

$$\phi(x) := H_k^{-1}(\sqrt{2\lambda}T(x)) \tag{2.3}$$

where

$$T(x) := \frac{b-a}{2} - \left| x - \frac{b+a}{2} \right|$$

and $k > 0$ is a solution to

$$\lambda = \frac{2}{(b-a)^2} H_k(G^{-1}(k))^2. \tag{2.4}$$

Moreover, all nonnegative solutions ϕ are in $C^2([a, b])$, and if the right derivative $g'_+(s) \to \infty$ as $s \to \infty$, then a nonnegative solution with $\lambda = \lambda^*$ exists.

Theorem 2.2 gives us both a method of computing λ^* and a method of finding explicit solutions to (2.1). Each solution to (2.4) gives a distinct solution to (2.1), so we may convert the problem of counting solutions to the differential equation (2.1) to finding solutions of the equation (2.4). We postpone the proof of lemma 2.1 and theorem 2.2 in favor of giving two examples that illustrate how the theorem can be used.

2.2 An example computing λ^*

Consider the case $g(s) := e^s$ and $\Omega := (0,1)$. We claim that $\lambda^* = 8L^2$, where $L \approx 0.66274$ is the Laplace limit constant, the unique real solution to the equation

$$\frac{L\exp(\sqrt{1+L^2})}{1+\sqrt{1+L^2}} = 1.$$

Numerical simulations show that $\lambda^* \approx 3.51383 \approx 8L^2$, and if we plot against $2H_k(G^{-1}(k))^2$ we find a convincing confirmation.

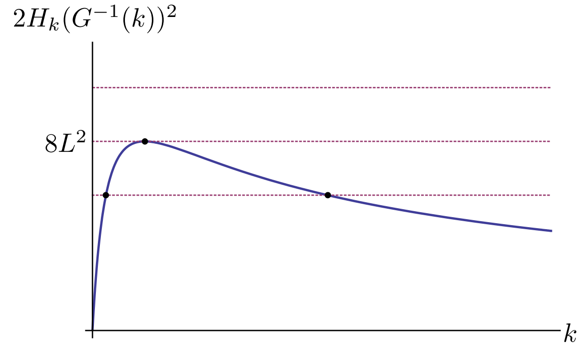

Figure 2.1: Counting solutions of $-\phi'' = \lambda e^{\phi}$. Each intersection corresponds to a solution of (2.1), two for the lowest line, exactly one for $8L^2$, and none above.

We proceed to prove what the figure suggests. We compute

$$G(u) = \int_0^u e^s\,ds = e^u - 1, \quad G^{-1}(k) = \log(k+1),$$

and

$$\begin{aligned} H_k(y) &= \int_0^y \frac{du}{\sqrt{k-(e^u-1)}} \\ &= \frac{2}{\sqrt{k+1}}\left(\operatorname{arctanh}\sqrt{\frac{k}{k+1}} - \operatorname{arctanh}\sqrt{\frac{k-(e^y-1)}{k+1}}\right) \end{aligned}$$

so

$$H_k(G^{-1}(k)) = \frac{2}{\sqrt{k+1}}\operatorname{arctanh}\sqrt{\frac{k}{k+1}}.$$

Differentiating in k gives

$$\frac{d}{dk}\left[H_k(G^{-1}(k))\right] = \frac{1}{(1+k)^{3/2}}\left(\sqrt{\frac{k+1}{k}} - \operatorname{arctanh}\sqrt{\frac{k}{k+1}}\right)$$

which is 0 when

$$\sqrt{\frac{k+1}{k}} = \operatorname{arctanh}\sqrt{\frac{k}{k+1}}.$$

Setting $w := \sqrt{\frac{k+1}{k}}$ we have $w = \coth w$ and $k = \frac{1}{w^2-1}$. We note that the equation $w = \coth w$ has a unique positive solution because $w \longmapsto w$ is increasing from 0 to ∞, while $w \longmapsto \coth w$ is decreasing. Thus we have uniquely determined a w and corresponding k. Plugging this k back in, we find

$$H_k(G^{-1}(k)) = \frac{2}{w\sqrt{k}} \operatorname{arctanh} \frac{1}{w} = \frac{2}{\sqrt{k}} = 2\sqrt{w^2-1}$$

so by theorem 2.2

$$\lambda^* = \frac{2}{1-0}(2\sqrt{w^2-1})^2 = 8(w^2-1).$$

In order that $\lambda^* = 8L^2$ we need that $L = \sqrt{w^2-1}$. We check that $\sqrt{w^2-1}$ satisfies the defining property of L,

$$\begin{aligned} 1 &= \frac{\sqrt{w^2-1}\exp(\sqrt{1+(\sqrt{w^2-1})^2})}{1+\sqrt{1+(\sqrt{w^2-1})^2}} \\ &\iff 1+w = \sqrt{w^2-1}\exp(w) \\ &\iff (w+1)^2 = (w^2-1)\exp(2w) \\ &\iff \exp(w)(w-1) = \exp(-w)(w+1) \\ &\iff \coth w = w \end{aligned}$$

which we know is true, hence our claim follows.

2.3 An example with no solution for $\lambda = \lambda^*$

In this section we show that it is possible for no nonnegative solution to (2.1) with $\lambda = \lambda^*$ to exist. Take $g(s) := 1 + 2s$ and $\Omega := (0, 1)$. We start with intuition and then use the theorem 2.2 to prove our claim rigorously. The ODE in question is

$$-\phi'' = \lambda(1 + 2\phi), \tag{2.5}$$

and the "general solution" of (2.5) is

$$\phi(x) = c_1 \cos(\sqrt{2\lambda}x) + c_2 \sin(\sqrt{2\lambda}x) - \frac{1}{2}.$$

Solving for the constants by applying boundary conditions gives

$$\phi(x) = \frac{1}{2}\Big(\cos(\sqrt{2\lambda}x) + (\csc(\sqrt{2\lambda}) - \cot(\sqrt{2\lambda}))\sin(\sqrt{2\lambda}x) - 1\Big) \tag{2.6}$$

from which it becomes apparent that problems begin occurring for the existence of nonnegative solutions whenever $\lambda \geq \frac{\pi^2}{2}$. Note that (2.6) is valid for λ not of the form $k^2\frac{\pi^2}{2}$ for some $k \in \mathbb{N}$ and it does give solutions to (2.1) for arbitrarily large λ. However, those solutions with $\lambda > \frac{\pi^2}{2}$ admit negative values. We plot these solutions for different values of λ in figure 2.3. As the figure shows, it appears that solutions to $-\phi'' = \lambda(1 + 2\phi)$ have fundamentally different behavior for $\lambda \in (k^2\frac{\pi^2}{2}, (k+1)^2\frac{\pi^2}{2})$ for different values of $k \in \mathbb{N}$. In particular, from the figure and from the

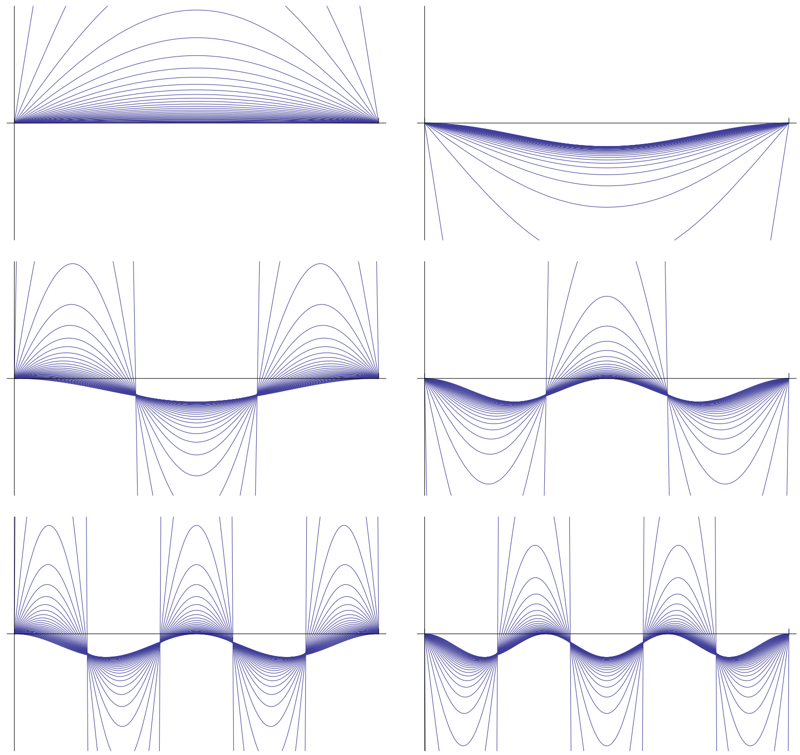

Figure 2.2: Sample solutions to $-\phi'' = \lambda(1+2\phi)$ plotted on $(0,1)$ with $\lambda \in (k^2\frac{\pi^2}{2}, (k+1)^2\frac{\pi^2}{2})$ for $k \in \{0,1,2,3,4,5\}$.

form of the solution (2.6) we see that nonnegative solutions blow up to ∞ as $\lambda \nearrow \frac{\pi^2}{2}$, so we expect that $\lambda^* = \frac{\pi^2}{2}$ and that there is no solution with $\lambda = \lambda^*$. This viewpoint is good for intuition, but not as strong as theorem 2.2. Since we do not necessarily have uniqueness of solutions to boundary value problems, even in the linear second order constant coefficients case, we cannot argue that (2.6) admitting negative values for some λ implies there are no nonnegative solutions for that λ, nor can we use the fact that the formula (2.6) doesn't make sense for $\lambda = \frac{\pi^2}{2}$ to conclude that there is no nonnegative solution for $\lambda = \frac{\pi^2}{2}$. However, theorem 2.2 is strong enough to imply these claims. We proceed to confirm our intuition by applying theorem 2.2 to show $\lambda^* = \frac{\pi^2}{2}$ and that there is no nonnegative solution with $\lambda = \lambda^*$. As in the previous subsection, we find after some calculation that

$$H_k(G^{-1}(k)) = \frac{\pi}{2} - \operatorname{arccot}(2\sqrt{k})$$

so that by theorem 2.2

$$\lambda^* = \frac{2}{1-0}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{2}.$$

However, $k \longmapsto H_k(G^{-1}(k))$ is strictly increasing to $\frac{\pi}{2}$, but it never attains the value $\frac{\pi}{2}$. Thus by the second part of theorem 2.2 there are no nonnegative solutions to (2.1) with $\lambda = \lambda^*$.

2.4 Proof of theorem 2.2

Proof of lemma 2.1. Suppose that ϕ solves (2.1). The setup of the problem implies that we have $\phi \in C([a,b]) \cap C^2((a,b))$. Then for $a < s < t < b$ we have

$$\phi'(t) = \phi'(s) + \int_s^t \lambda g(\phi(r))\,dr.$$

Since $g \circ \phi$ is continuous on $[a,b]$ it follows that $\phi'(t) \to \ell$ as $t \to b^-$ for some $\ell \in \mathbb{R}$. Then L'Hôpital's rule implies ϕ is differentiable at b with $\phi'(b) = \ell$. In turn,

$$\frac{\phi'(b) - \phi'(s)}{b-s} = \frac{1}{b-s}\int_s^b \lambda g(\phi(r))\,dr \to \lambda g(\phi(b))$$

as $s \to b^-$, so that $\phi''(b) = \lambda g(\phi(b))$. We apply a similar argument for $\phi''(a)$ to show $\phi \in C^2([a,b])$ and that $-\phi'' = \lambda g(\phi)$ holds on all of $[a,b]$. If $g \in C^k([0,\infty))$ we may then bootstrap the fact that $\phi \in C^{k+2}([a,b])$ by observing that $\phi \in C^n([a,b])$ and $\phi'' = -\lambda g(\phi)$ in $[a,b]$ implies $\phi \in C^{n+2}([a,b])$ for any $n \in \mathbb{N}$ with $n \leq k$. □

Proof of theorem 2.2. Suppose that ϕ is a nonnegative solution of (2.1). By lemma 2.1 we know that $\phi \in C^2([a,b])$ and $-\phi'' = \lambda g(\phi)$ on $[a,b]$. Multiplying (2.1) by ϕ' we find

$$\frac{d}{ds}\left[-\frac{1}{2}\phi'(s)^2\right] = \frac{d}{ds}\left[\lambda G(\phi(s))\right].$$

Integrating from a to $x \in [a,b]$ then gives

$$-\frac{1}{2}(\phi'(x)^2 - \phi'(a)^2) = \lambda(G(\phi(x)) - G(\phi(a))) = \lambda G(\phi(x))$$

where here we imposed the boundary condition $\phi(a) = 0$ along with $G(0) = 0$. Rearranging and solving for $\phi'(x)$ we find that for every $x \in [a,b]$

$$\phi'(x) = \sqrt{\phi'(a)^2 - 2\lambda G(\phi(x))} \text{ or } \phi'(x) = -\sqrt{\phi'(a)^2 - 2\lambda G(\phi(x))}$$

where the choice potentially depends on x. We know that $\phi'' = -\lambda g(\phi) < 0$ so it follows that $\phi'(a) > 0$ since $\phi \geq 0$. Similarly $\phi'(b) < 0$ and so ϕ's maximum occurs in (a,b) and is unique. Call $x^* \in (a,b)$ the unique point such that $\max_{x\in[a,b]} \phi(x) = \phi(x^*)$. Thus the sign of ϕ' does depend on x,

$$\phi'(x) = \begin{cases} \sqrt{\phi'(a)^2 - 2\lambda G(\phi(x))} & x \in [a, x^*], \\ -\sqrt{\phi'(a)^2 - 2\lambda G(\phi(x))} & x \in [x^*, b] \end{cases} \tag{2.7}$$

for all $x \in [a,b]$. Plugging in x^* tells us that

$$\max_{x\in[a,b]} \phi(x) = G^{-1}\left(\frac{\phi'(a)^2}{2\lambda}\right). \tag{2.8}$$

Continuing from (2.7), for $s \neq x^*$ we have $\sqrt{\phi'(a)^2 - 2\lambda G(\phi(s))} \neq 0$ and may divide to find

$$\frac{\phi'(s)}{\sqrt{\phi'(a)^2 - 2\lambda G(\phi(s))}} = 1_{[a,x^*]}(s) - 1_{[x^*,b]}(s). \tag{2.9}$$

Integrating over $[a,b]$ gives

$$0 = (x^* - a) - (b - x^*)$$

so that $x^* = \frac{a+b}{2}$ as expected. Recall that $T(x) = \frac{b-a}{2} - \left|x - \frac{b+a}{2}\right|$ is the triangular bump with $T(a) = T(b) = 0$ and $T(x^*) = T\left(\frac{b+a}{2}\right) = \frac{b-a}{2}$. Define the temporarily useful

$$F(y) := \int_0^y \frac{du}{\sqrt{\phi'(a)^2 - 2\lambda G(u)}}.$$

We may integrate (2.9) from a to $x \in [a,b]$ which gives

$$F(\phi(x)) = \int_0^{\phi(x)} \frac{du}{\sqrt{\phi'(a)^2 - 2\lambda G(u)}} = T(x). \tag{2.10}$$

Since $g \in C([0,\infty))$ we know $G \in C^1([0,\infty))$, so that $F \in C^2([0, G^{-1}(\frac{\phi'(a)^2}{2\lambda})))$ with $F' > 0$. Thus F is invertible and so for $k := \frac{\phi'(a)^2}{2\lambda}$

$$\phi(x) = F^{-1}(T(x)) = H_k^{-1}(\sqrt{2\lambda}T(x)) \tag{2.11}$$

must be the form of the solution. Then by (2.8) and (2.10) we have that

$$H_k(G^{-1}(k)) = H_k(\phi(x^*)) = \sqrt{2\lambda}F(\phi(x^*)) = \sqrt{2\lambda}\frac{b-a}{2}$$

so that indeed (2.4) is satisfied.

Now that we know what the form of the solution must be, we attempt to verify when (2.11) actually solves (2.1). Suppose we use (2.11) to define ϕ. We have implicitly fixed $\phi'(a)$, which appears in the formula for F, so we cannot use this to define ϕ. However, by replacing $\phi'(a)$ with a constant $c > 0$ in the formula, we may use (2.11) as a definition. With this definition of ϕ, we indeed find by the inverse function theorem that $\phi'(a) = \frac{1}{F'(F^{-1}(0))} = c$ so there is no ambiguity. That is, we have already shown that the form of ϕ must be (2.11), and we need only verify that if k solves (2.4), then the choice $\phi'(a) := \sqrt{2\lambda}$ produces a solution of (2.1).

Fix a k solving (2.4). Since F^{-1} is increasing, we know the maximum of ϕ occurs at the same maximum $x^* = \frac{b+a}{2}$ of T, i.e.

$$\max_{x\in[a,b]} \phi(x) = H_k^{-1}\left(\sqrt{2\lambda}\frac{b-a}{2}\right) = H_k^{-1}(H_k(G^{-1}(k))) = G^{-1}(k)$$

where in the second equality we used (2.4). We quickly note that

$$\begin{aligned} H_k(G^{-1}(k)) &= \int_0^{G^{-1}(k)} \frac{du}{\sqrt{k - G(u)}} \\ &\le \int_0^{G^{-1}(k)} \frac{ds}{\sqrt{k - \frac{k}{G^{-1}(k)}s}} \\ &= \frac{2G^{-1}(k)}{\sqrt{k}} \end{aligned} \tag{2.12}$$

by convexity of G, which follows from the fact that $G' = g$ is continuous and increasing. We note that the contents of the square root are indeed positive in the previous line for $s < G^{-1}(k)$. So in fact F is bounded and therefore continuous at $G^{-1}(k)$ by the monotone convergence theorem. Hence F^{-1} is continuous at $F(G^{-1}(k)) = \frac{b-a}{2}$. Moreover, the inverse function theorem gives us that $F^{-1} \in C^2([0, \frac{b-a}{2}))$ and so we immediately have that ϕ is C^2 except possibly at x^*. But

$$F'(u) = \frac{1}{\phi'(a)^2 - 2\lambda G(u)} \to \infty$$

as $u \nearrow G^{-1}(k)$, so by the mean value theorem

$$\begin{aligned}
(F^{-1})'\left(\frac{b-a}{2}\right) &= \lim_{y \to \frac{b-a}{2}} \frac{F^{-1}(\frac{b-a}{2}) - F^{-1}(y)}{\frac{b-a}{2} - y} \\
&= \lim_{x \to G^{-1}(k)} \frac{G^{-1}(k) - x}{F(G^{-1}(k)) - F(x)} \\
&= \lim_{x \to G^{-1}(k)} \frac{1}{F'(\xi_x)} \\
&= 0
\end{aligned}$$

where ξ_x is provided by the mean value theorem. This tells us that we have $F^{-1} \in C^1([0, \frac{b-a}{2}])$. We use a similar argument for $(F^{-1})''$, by L'Hôpital's rule we have

$$\begin{aligned}
(F^{-1})''\left(\frac{b-a}{2}\right) &= \lim_{y \to \frac{b-a}{2}} \frac{(F^{-1})'(\frac{b-a}{2}) - (F^{-1}(y))'}{\frac{b-a}{2} - y} \\
&= \lim_{x \to G^{-1}(k)} \frac{0 - \frac{1}{F'(x)}}{F(G^{-1}(k)) - F(x)} \\
&\overset{\text{L'H}}{=} \lim_{x \to G^{-1}(k)} \frac{-F''(x)/F'(x)^2}{F'(x)} \\
&= \lim_{x \to G^{-1}(k)} \frac{-F''(x)}{F'(x)^3} \\
&= \lim_{x \to G^{-1}(k)} \frac{-\lambda g(x)(\phi'(a)^2 - 2\lambda G(x))^{-3/2}}{(\phi'(a)^2 - 2\lambda G(x))^{-3/2}} \\
&= -\lambda g(G^{-1}(k)).
\end{aligned}$$

A simple calculation from the inverse function theorem shows that $(F^{-1})''(x) \to -\lambda g(G^{-1}(k))$ as $x \nearrow G^{-1}(k)$, and therefore we also have $F^{-1} \in C^2([0, \frac{b-a}{2}])$. Since we know F^{-1} is regular enough on its domain, we are now equipped to show $\phi = F^{-1} \circ T$ is C^2 at x^*. Restricting to $[a, x^*]$ or to $[x^*, b]$ gives us that $\phi \in C^2([a, x^*]) \cap C^2([x^*, b])$, so we need only check that the left and right hand limits of derivatives match in order to show $\phi \in C^2([a, b])$. We have

$$\phi'(x) = (F^{-1})'(T(x))T'(x) \to 0$$

as $x \to x^*$ since $T(x) \to T(x^*) = \frac{b-a}{2}$ as $x \to x^*$ and $|T'(x)| = 1$ for $x \neq x^*$. Furthermore,

$$\begin{aligned}\phi''(x) &= (F^{-1})''(T(x))(T'(x))^2 + (F^{-1})'(T(x))T''(x) \\ &= (F^{-1})''(T(x)) \\ &\to (F^{-1})''\left(\frac{b-a}{2}\right)\end{aligned}$$

as $x \to x^*$ since $|T'(x)| = 1$ and $T''(x) = 0$ for all $x \neq x^*$. Thus we have shown $\phi \in C^2([a, b])$. Easy calculations then verify that $-\phi'' = \lambda g(\phi)$ on $[a, b]$.

At this point, we have completely classified solutions to (2.1) in terms of solutions to (2.4). From this correspondence, we claim that we can directly see the existence of and formula for the critical λ^*. Define $M(k) := H_k(G^{-1}(k))$ for $k > 0$ and $M(0) := 0$. We will show that M is continuous and bounded on $[0, \infty)$. This will imply $M([0, \infty))$ is a bounded interval containing 0, which implies the existence of λ^*.

First we show M is continuous on $[0, \infty)$. From above in (2.12) we know that

$$M(k) \leq \frac{2G^{-1}(k)}{\sqrt{k}} =: B(k).$$

We consider what happens to $B(k)$ when $k \searrow 0$. Applying L'Hôpital's rule, we have

$$\begin{aligned}\lim_{k\to 0} \frac{2G^{-1}(k)}{\sqrt{k}} &= \lim_{u\to 0} \frac{2u}{\sqrt{G(u)}} \\ &= 2\sqrt{\lim_{u\to 0} \frac{u^2}{G(u)}} \\ &\overset{\text{L'H}}{=} 2\sqrt{\lim_{u\to 0} \frac{2u}{g(u)}} \\ &= 2\sqrt{\frac{2\cdot 0}{g(0)}} \\ &= 0.\end{aligned}$$

It follows that M is continuous at 0. Furthermore, for $k \in (0, \infty)$ the integrand in $M(k)$ is dominated,

$$\frac{1_{s\leq G^{-1}(k)}}{\sqrt{k - G(s)}} \leq \frac{1_{s\leq G^{-1}(k)}}{\sqrt{k - \frac{k}{G^{-1}(k)}s}},$$

where the integral of the right side, $B(k)$, satisfies dominated convergence as $k \to k_0$ for any $k_0 \in (0, \infty)$. It follows by dominated convergence that M is continuous on $(0, \infty)$. Hence $\{k : M(k) = \lambda\}$ is closed and bounded below by 0, so it has a minimum if it is nonempty. In the case that the minimum exists for a given λ, it cannot be 0 since $0 < \lambda$ so that $M(k) < \lambda/2 < \lambda$ in a neighborhood of 0. For a fixed λ, sending $k \searrow 0$ amounts to sending $\phi'(a) \searrow 0$. Since from (2.11) we see that $\phi(x)$ is strictly increasing in $\phi'(a)$, the smallest valid k will give a minimal solution. Thus, if there is a solution to (2.1), there is a minimal positive solution to (2.1).

Next we consider what happens to our bound $B(k)$ as $k \to \infty$. Since g is increasing, convex, and not a constant, there exists an $s_0 \geq 0$ such that $g'_+(s_0) > 0$, where g'_+ denotes the right derivative of g. Then

$$\begin{aligned}
\limsup_{k\to\infty} \frac{2G^{-1}(k)}{\sqrt{k}} &= \limsup_{u\to\infty} \frac{2u}{\sqrt{G(u)}} \\
&= \limsup_{u\to\infty} \frac{2u}{\sqrt{\int_0^u g(s)\,ds}} \\
&\leq \limsup_{u\to\infty} \frac{2u}{\sqrt{\int_0^u [g(s_0) + (s-s_0)g'_+(s_0)]\,ds}} \\
&= \limsup_{u\to\infty} \frac{2u}{\sqrt{u[g(s_0) - s_0 g'_+(s_0)] + \frac{u^2}{2} g'(s_0)}} \\
&= \frac{2}{\sqrt{\frac{1}{2} g'_+(s_0)}} \qquad (2.13) \\
&< \infty
\end{aligned}$$

which shows that $B(k)$ stays bounded as $k \to \infty$.

Thus the critical λ^* exists, and moreover we have a formula for it:

$$\lambda^* = \sup_{k>0} \left[\frac{2}{(b-a)^2} H_k (G^{-1}(k))^2 \right].$$

If we assume that $g'_+(u) \to \infty$ as $u \to \infty$ we can conclude by (2.13) that $M(k) \to 0$ as $k \to \infty$ as well. In this case, we know that the supremum for λ^* is not attained by sending $k \to \infty$, and it follows that the supremum is actually a maximum and solution with $\lambda = \lambda^*$ exists. □

3 The d-dimensional case $-\triangle\phi + u \cdot \nabla\phi = \lambda g(\phi)$

3.1 Improving known bounds on $\lambda^*(u)$

We concern ourselves with the problem

$$\begin{cases} -\triangle\phi + u \cdot \nabla\phi = \lambda g(\phi) & x \in \Omega \\ \phi = 0 & x \in \partial\Omega. \end{cases} \tag{3.1}$$

In this section $\Omega \subseteq \mathbb{R}^d$ is a nonempty, open, bounded, connected domain with $C^{2,\alpha}$ boundary for some $\alpha \in (0,1)$. We also assume $u : \overline{\Omega} \to \mathbb{R}^d$ is Lipschitz continuous, and $g : [0,\infty) \to (0,\infty)$ is increasing, convex, and twice differentiable with $g'(0) > 0$ and $g'(s) \to \infty$ as $s \to \infty$.

We will show the existence of the explosion threshold $\lambda^*(u)$, as well as give lower and upper bounds for it. The proofs in this section are largely inspired by those in [4] and [31]. We state the results first for readability and give proofs in the following section. We start with what will turn out to be a lower bound for $\lambda^*(u)$.

Proposition 3.1. *Let θ_u be the maximum of the solution to (3.1) with $\lambda g = 1$. Then (3.1) admits a minimal nonnegative solution ϕ for all $\lambda < \frac{1}{\theta_u} \sup_{B \geq 0} \frac{B}{g(B)}$. Moreover, for any $B > 0$, if additionally $\lambda \leq \frac{1}{\theta_u} \frac{B}{g(B)}$, then the minimal solution satisfies $\|\phi\|_\infty \leq B$.*

It is clear that $\sup_{B \geq 0} \frac{B}{g(B)} \in (0,\infty)$ since $\frac{B}{g(B)} \to 0$ as $B \to \infty$ by L'Hôpital's rule. At the moment, we do not know whether the explosion threshold exists. We work towards showing it exists by proving that the set of $\lambda > 0$ for which there is a solution to (3.1) is an interval.

Lemma 3.2. *If (3.1) admits a nonnegative solution for some $\lambda_0 > 0$, then it also admits a nonnegative solution for all $\lambda \in (0, \lambda_0)$.*

To finish proving the existence of $\lambda^*(u) \in (0,\infty)$, we need an upper bound of the set of $\lambda > 0$ for which (3.1) admits a nonnegative solution. We will find the upper bound by using a special value K for which $g(K) - Kg'(K) = 0$ along with the principal eigenvalue of an adjoint problem.

Lemma 3.3. *The function $h(k) := g(k) - kg'(k)$ is decreasing and there is a unique $K > 0$ such that $g(K) - Kg'(K) = 0$ and $g(k) - kg'(k) > 0$ for all $k < K$. Moreover, $\sup_{B \geq 0} \frac{B}{g(B)} = \frac{K}{g(K)} = \frac{1}{g'(K)}$.*

Let $\mu_1[u]$ and η respectively denote the principle eigenvalue and positive eigenfunction, normalized so that $\int_\Omega \eta \, dx = 1$, of the adjoint problem

$$\begin{cases} -\triangle\eta - \nabla \cdot (u\eta) = \mu_1[u]\eta & x \in \Omega \\ \eta = 0 & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Existence of $\eta \in W^{2,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ for all $p < \infty$ for a general domain Ω was established in [5]. Also note that $-\triangle + u \cdot \nabla$ has no zero order term, so that its principal eigenvalue is real and strictly positive, cf. [5]. The real eigenvalues of $-\triangle + u \cdot \nabla$ and its formal adjoint $-\triangle - u \cdot \nabla - \nabla \cdot u$ are identical, cf. theorem 8.6 in [14], so it follows that $\mu_1[u] > 0$.

Proposition 3.4. *Let $K > 0$ be given by lemma 3.3 and let $\mu_1[u]$ and η respectively be the principle eigenvalue and normalized positive eigenfunction of the adjoint problem* (3.2)*. If ϕ is any solution to* (3.1)*, then for every $p \in [1, \infty]$ we have*

$$\|\phi\|_p \geq \frac{\lambda}{\|\eta\|_{p'}} \sup_{k<K} \frac{g(k) - kg'(k)}{\mu_1[u] - \lambda g'(k)}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, for all

$$\lambda > \mu_1[u] \sup_{B \geq 0} \frac{B}{g(B)} = \mu_1[u] \frac{K}{g(K)} = \mu_1[u] \frac{1}{g'(K)}$$

the problem (3.1) *admits no nonnegative solutions.*

Putting together the previous results then gives us the existence of and bounds on the explosion threshold for the problem (3.1).

Theorem 3.5. *The explosion threshold $\lambda^*(u)$ for the problem* (3.1) *exists and satisfies*

$$\frac{1}{\theta_u} \sup_{B \geq 0} \frac{B}{g(B)} \leq \lambda^*(u) \leq \mu_1[u] \sup_{B \geq 0} \frac{B}{g(B)}$$

where θ_u is the maximum of the solution to (3.1) *with $\lambda g = 1$, and $\mu_1[u]$ is the principal eigenvalue of the adjoint problem* (3.2)*.* □

Remark 3.6. It would be nice if we had $\frac{1}{\theta_u} = \mu_1[u]$ as this would make the bounds in theorem 3.5 coincide and hence give a formula for $\lambda^*(u)$. However, this equality never holds. Indeed, if we choose τ, ξ so that $-\triangle\tau + u \cdot \nabla\tau = 1$ and $-\triangle\xi + u \cdot \nabla\xi = \mu_1[u]\xi$ in Ω, both with zero boundary conditions, then $q := \tau - \xi\frac{1}{\mu_1[u]\|\xi\|_\infty}$ is continuous and satisfies $-\triangle q + u \cdot \nabla q \geq 0$ in Ω and $q = 0$ on $\partial\Omega$. The (distributional) strong maximum principle, e.g. theorem 8.19 in [14], then implies $q \geq 0$ and $q = 0$ somewhere in Ω only if q is constant. It cannot be that q is constant, else $\tau = \xi$ and so $1 = \mu_1[u]\xi$. That is, ξ would have to be constant $\frac{1}{\mu_1[u]}$ which violates the fact that $\xi = 0$ on $\partial\Omega$. Thus $q > 0$ on Ω and so plugging in an $x \in \Omega$ such that $\xi(x) = \|\xi\|_\infty$ tells us $\frac{1}{\theta_u} \leq \frac{1}{\tau(x)} < \mu_1[u]\frac{\|\xi\|_\infty}{\xi(x)} = \mu_1[u]$.

Remark 3.7. Applying the bounds in theorem 3.5 to the $\lambda^* = 8L^2$ computed in section 2.2 shows that neither bound in theorem 3.5 is sharp. In that example the bounds read

$$\frac{8}{e} \leq 8L^2 \leq \frac{\pi^2}{e}$$

or approximately

$$2.94304 \leq 3.51383 \leq 3.63082.$$

Remark 3.8. Our proofs of the results in this sections will not require u to be divergence free, but if we restrict our attention to divergence free u, then we may bound θ_u uniformly among advecting flows u, cf. [18] or lemma 4.1, so $\lambda^*(u)$ may be bounded below independently of u.

3.2 Proofs of $\lambda^*(u)$ bounds

To prove proposition 3.1 and lemma 3.2, we will make use of a stochastic representation of solutions to the Dirichlet problem. The reader unfamiliar with this representation should see section 5.1 in the appendix for a brief introduction.

Proof of proposition 3.1. Fix $0 < \lambda < \frac{1}{\theta_u} \sup_{B \geq 0} \frac{B}{g(B)}$ and choose $B > 0$ so that $\lambda < \frac{B}{g(B)}$. We will define a sequence of approximating subsolutions which increase to a true solution. Let $\phi^{(0)} := 0$ and recursively choose $\phi^{(n+1)}$ as the solution to

$$-\triangle \phi^{(n+1)} + u \cdot \nabla \phi^{(n+1)} = \lambda g(\phi^{(n)}) \tag{3.3}$$

in Ω with $\phi^{(n)} = 0$ on $\partial\Omega$. Our assumptions on Ω, g, and u suffice in order to prove the existence of the iterates $\phi^{(n)} \in C^{2,\alpha}(\overline{\Omega})$, see theorem 6.14 in [14]. Then for each $n \in \mathbb{N}$ the stochastic representation theorem 5.1 implies

$$\phi^{(n+1)}(x) = E^x \int_0^{\tau_\Omega} \lambda g(\phi^{(n)}(X_t))\, dt$$

for all $x \in \overline{\Omega}$ for a suitable stochastic diffusion X and exit time τ_Ω. By induction, $\phi^{(n)} \geq 0$ and

$$\phi^{(n+1)}(x) - \phi^{(n)}(x) = \lambda E^x \int_0^{\tau_\Omega} \left[g(\phi^{(n)}(X_t)) - g(\phi^{(n-1)}(X_t)) \right] dt \geq 0$$

since g is increasing. It follows that $\phi^{(n)}$ is increasing in n. Also by induction, we have that $\phi^{(n)} \leq b_n$ where $b_0 := 0$ and $b_{n+1} := \lambda g(b_n)\theta_u$. To see the induction step, note

$$\begin{aligned}
\phi^{(n+1)}(x) &= E^x \int_0^{\tau_\Omega} \lambda g(\phi^{(n)}(X_t))\, dt \\
&\leq \lambda g(b_n) E^x \tau_\Omega \\
&\leq \lambda g(b_n)\theta_u \\
&= b_{n+1}.
\end{aligned}$$

Here we used the fact that

$$\theta_u = \max_{x \in \overline{\Omega}} E^x \tau_\Omega = \max_{x \in \overline{\Omega}} E^x \int_0^{\tau_\Omega} 1\, dt.$$

Since the $\phi^{(n)}$ are increasing in n, it suffices to show they are bounded above to show that their limit $\phi := \lim_{n\to\infty} \phi^{(n)}$ is finite. We will show the b_n are bounded. Certainly $b_0 = 0 < B$ and by induction and the fact that $\lambda < \frac{1}{\theta_u} \frac{B}{g(B)}$ we have

$$b_{n+1} = \lambda g(b_n)\theta_u \leq \lambda g(B)\theta_u \leq B.$$

Thus we have shown that $\phi^{(n)} \nearrow \phi \leq B$. Note that all we needed to conclude this was $\lambda \leq \frac{B}{g(B)\theta_u}$, and so if $\lambda \leq \frac{B'}{g(B')\theta_u}$ for some smaller B', we can conclude $\|\phi\|_\infty \leq B'$.

Once we have that $\left\{\phi^{(n)}\right\}_n$ is a uniformly bounded and pointwise convergent sequence, we may essentially repeat the proofs of theorem 2.1 and corollary 2.2 in [31] to show that ϕ is the minimal positive solution to (3.1). □

Proof of lemma 3.2. Fix $\lambda < \lambda_0$. Repeat the proof of proposition 3.1, but instead of using the explicit hypothesis about how small λ is to get the bound $\phi^{(n)} \leq B$, use that $\phi^{(n)} \leq \phi_{\lambda_0}$ for all n, where ϕ_{λ_0} is the minimal positive solution for (3.1) when $\lambda = \lambda_0$. To see that $\phi^{(n)} \leq \phi_{\lambda_0}$, simply note that $0 = \phi^{(0)} \leq \phi_{\lambda_0}$ and by induction

$$\begin{aligned}\phi_{\lambda_0}(x) - \phi^{(n+1)}(x) &= E^x \int_0^{\tau_\Omega} \left[\lambda_0 g(\phi_{\lambda_0}(X_t)) - \lambda g(\phi^{(n)}(X_t))\right] dt \\ &\geq E^x \int_0^{\tau_\Omega} \lambda \left[g(\phi_{\lambda_0}(X_t)) - g(\phi^{(n)}(X_t))\right] dt \\ &\geq 0\end{aligned}$$

where in the last step we used the induction hypothesis $\phi^{(n)} \leq \phi_{\lambda_0}$. Then $\left\{\phi^{(n)}\right\}_n$ is pointwise convergent and uniformly bounded by $\max_{x\in\overline{\Omega}} \phi_{\lambda_0}$, so the rest of the proof goes through. □

Proof of lemma 3.3. To see that h is decreasing note that $h'(k) = -kg''(k) \leq 0$. Assume for contradiction that $h(k) = g(k) - kg'(k) \geq 0$ for all $k \geq 0$. Then $\frac{g'(k)}{g(k)} \leq \frac{1}{k}$ for all $k > 0$. Integrating then gives $\log g(k) \leq \log g(1) + \log k$ and hence $\frac{g(k)}{k} \leq g(1)$ for $k > 0$. Since $g(k) \to \infty$ as $k \to \infty$ we are in a position to apply L'Hôpital's rule to find

$$g(1) \geq \lim_{k\to\infty} \frac{g(k)}{k} \overset{\text{L'H}}{=} \lim_{k\to\infty} g'(k) = \infty$$

which gives a contradiction. Hence it must be that $h(k) < 0$ for some $k > 0$. Since $h(0) = g(0) - 0g'(0) > 0$, the intermediate value theorem gives the existence of K.

Next we show that $\sup_{B\geq 0} \frac{B}{g(B)} = \frac{K}{g(K)} = \frac{1}{g'(K)}$. The last equality follows immediately from rearranging $g(K) - Kg'(K) = 0$, so we focus on the first. We know $\frac{B}{g(B)} \to 0$ as $B \to 0$ and as $B \to \infty$ by L'Hôpital's rule. Thus the supremum occurs at an interior point where the derivative is zero. Calculating the derivative we find

$$\frac{d}{dB}\left[\frac{B}{g(B)}\right] = \frac{g(B) - Bg'(B)}{g(B)^2} = \frac{h(B)}{g(B)^2} = 0$$

only when $h(B) = 0$. We have already defined K to be the first value for which $h = 0$. Let K_2 denote the last value for which $h = 0$. There is a last value since we already showed h is decreasing and admits negative values. By what we have already shown $h > 0$ on $[0, K)$, $h = 0$ on $[K, K_2]$, and $h < 0$ on (K_2, ∞). It follows that $B \longmapsto \frac{B}{g(B)}$ is increasing, constant, and decreasing on these regions respectively, so it follows that $\sup_{B\geq 0} \frac{B}{g(B)} = \frac{K}{g(K)}$. □

Proof of proposition 3.4. If ϕ solves (3.2), then

$$\begin{aligned}\mu_1[u] \int_\Omega \eta\phi\, dx &= \int_\Omega \phi[-\triangle\eta + \nabla\cdot(u\eta)]\, dx \\ &= \int_\Omega \eta[-\triangle\phi + u\cdot\nabla\phi]\, dx \\ &= \int_\Omega \lambda\eta g(\phi)\, dx\end{aligned}$$

where here we used integration by parts multiple times in conjunction with the zero boundary conditions. Note that since g is convex we have that $g(s) \geq g(k) + g'(k)(s-k)$ for all $s, k \geq 0$. Combining this with the fact that $\int_\Omega \eta \, dx = 1$, for any $k \geq 0$ we have

$$\begin{aligned} \mu_1[u] \int_\Omega \eta\phi \, dx &= \int_\Omega \lambda \eta g(\phi) \, dx \\ &\geq \int_\Omega \lambda \eta [g(k) + g'(k)(\phi - k)] \, dx \\ &= \lambda \left[g(k) - kg'(k) + g'(k) \int_\Omega \eta\phi \, dx \right]. \end{aligned} \tag{3.4}$$

Recall $K > 0$ is the unique value, which we know exists by lemma 3.3, such that $g(K) - Kg'(K) = 0$ and that $g(k) - kg'(k) > 0$ for $k < K$. We note that the right side of (3.4) is strictly positive for all $k \leq K$, so that we are justified in preserving the inequality when dividing to find

$$\lambda \leq \frac{\mu_1[u] \int_\Omega \eta\phi \, dx}{g(k) - kg'(k) + g'(k) \int_\Omega \eta\phi \, dx} \tag{3.5}$$

for all $k \leq K$. Then for $k < K$ we find

$$\lambda < \mu_1[u] \frac{1}{g'(k)} \tag{3.6}$$

and for $k = K$ we find the desired bound

$$\lambda \leq \mu_1[u] \frac{1}{g'(K)} = \mu_1[u] \frac{K}{g(K)} = \mu_1[u] \sup_{B \geq 0} \frac{B}{g(B)}$$

where the last two equalities are provided by lemma 3.3. Rearranging (3.4) we find that for $k \geq 0$, we have

$$(\mu_1[u] - \lambda g'(k)) \int_\Omega \eta\phi \, dx \geq \lambda (g(k) - kg'(k)).$$

For $k < K$ we have $\lambda < \frac{\mu_1[u]}{g'(k)}$ by (3.6), so we may apply Hölder's inequality and divide to say that

$$\|\phi\|_p \|\eta\|_{p'} \geq \frac{\lambda (g(k) - kg'(k))}{\mu_1[u] - \lambda g'(k)}$$

for any conjugate $p, p' \in [1, \infty]$. Rearranging and taking the supremum over $k < K$ finally gives

$$\|\phi\|_p \geq \frac{\lambda}{\|\eta\|_{p'}} \sup_{k < K} \frac{g(k) - kg'(k)}{\mu_1[u] - \lambda g'(k)}$$

which completes the proof. □

4 Reducing two dimensions to one dimension

As theorem 2.2 allows us to compute the explosion threshold in one dimension, we would like to reduce the d-dimensional problem to the one-dimensional problem. Some progress in this direction was made by Iyer et. al. in [18]. They used fast-flow asymptotics and variational methods to reduce the two-dimensional case of

$$\begin{cases} -\triangle \phi^{Au} + Au \cdot \nabla \phi^{Au} = 1 & x \in \Omega \\ \phi^{Au} = 0 & x \in \partial\Omega \end{cases}$$

to a one-dimensional Friedlin problem of the form

$$\begin{cases} -\dfrac{1}{T(h)} \dfrac{d}{dh} \left(p(h) \dfrac{df}{dh} \right) = \overline{g}(h) \\ \\ f(0) = 0, f(h) \text{ is bounded for } 0 \leq h \leq M. \end{cases}$$

We attempted to extend the solved case of $-\phi'' = \lambda g(\phi)$ to allow for nonconstant coefficients. Ultimately, we were only able to extend to the case $p(h) = 1/T(h)$, which did not turn out to be useful in calculating the explosion threshold in two dimensions. Along the way we wrote down a PDE proof of a well-known fast-flow convergence in two dimensions.

4.1 Allowing nonconstant coefficients

We can make a slight generalization to allow more general operators on the left side of (2.1). Indeed, if ϕ satisfies (2.1) and $\varphi : [c, d] \to [a, b]$ is C^2 and satisfies $\varphi(c) = a, \varphi(d) = b$, then

$$\frac{-1}{\varphi'} \left(\frac{1}{\varphi'} (\phi \circ \varphi)' \right)' = \frac{-1}{\varphi'} (\phi' \circ \varphi)' = -\phi'' \circ \varphi = \lambda g(\phi \circ \varphi)$$

so $r := \phi \circ \varphi$ satisfies the more general problem

$$\begin{cases} -\dfrac{1}{\varphi'} \dfrac{d}{dx} \left(\dfrac{1}{\varphi'} \dfrac{dr}{dx} \right) = \lambda g(r) & x \in (c, d) \\ r = 0 & x \in \{c, d\}. \end{cases} \tag{4.1}$$

If we additionally have that φ is invertible with C^2 inverse, then this process is reversible. Assuming r satisfies (4.1) we check

$$\begin{aligned} -(r \circ \varphi^{-1})'' &= -\left(\frac{r' \circ \varphi^{-1}}{\varphi' \circ \varphi^{-1}} \right)' \\ &= -\frac{(r' \circ \varphi^{-1})'(\varphi' \circ \varphi^{-1}) - (r' \circ \varphi^{-1})(\varphi' \circ \varphi^{-1})'}{(\varphi' \circ \varphi^{-1})^2} \\ &= -\frac{\frac{r'' \circ \varphi^{-1}}{\varphi' \circ \varphi^{-1}} (\varphi' \circ \varphi^{-1}) - (r' \circ \varphi^{-1}) \frac{\varphi'' \circ \varphi^{-1}}{\varphi' \circ \varphi^{-1}}}{(\varphi' \circ \varphi^{-1})^2} \\ &= \left[-\frac{1}{\varphi'} \left(\frac{1}{\varphi'} r' \right)' \right] \circ \varphi^{-1} \\ &= \lambda (g \circ r) \circ \varphi^{-1} \\ &= \lambda g \circ (r \circ \varphi^{-1}) \end{aligned}$$

so $r \circ \varphi^{-1}$ satisfies (2.1). It follows that, under our smoothness and invertibility assumptions on φ, there is a solution to (2.1) if and only if there is a solution to (4.1). This immediately gives us the existence of a critical λ^* for the problem (4.1) and tells us that it coincides with the λ^* given in (2.2), i.e. the λ^* that we computed explicitly with φ replaced by the identity function.

4.2 Fast-flow convergence in two dimensions

We consider the family of Poisson problems

$$\begin{cases} -\triangle\phi^{Au} + Au\cdot\nabla\phi^{Au} = g(x) & x\in\Omega \\ \phi^{Au} = 0 & x\in\partial\Omega \end{cases} \tag{4.2}$$

for $A \neq 0$. In this section $\Omega \subseteq \mathbb{R}^2$ is nonempty, open, bounded, simply connected, and has C^2 boundary. Also $u : \overline{\Omega} \to \mathbb{R}^2$ is an incompressible $C^1(\overline{\Omega})$ flow tangential to $\partial\Omega$. Let $\psi \in C^2(\overline{\Omega})$ be the stream function of u, i.e. $\psi = 0$ on $\partial\Omega$ and $u = \nabla^{\perp}\psi = (-\partial_2\psi, \partial_1\psi)$. We further assume that ψ has a single nondegenerate critical point $x_0 \in \Omega$. We may assume without loss of generality that ψ achieves a maximum M at x_0. In the case that ψ achieves a minimum at x_0, we replace u with $-u$ and A with $-A$. Define $\Omega_{\psi,h} := \{x \in \Omega : \psi(x) > h\}$ to be the h-super-level set of ψ. Note that every $h \in (0, M)$ is a regular value of ψ, $\Omega_{\psi,h}$ is simply connected, and $\partial\Omega_{\psi,h}$ is a C^2 Jordan curve by the implicit function theorem.

We claim that in this situation the ϕ^{Au} converge uniformly to a function that is constant along stirring lines, i.e. the streamlines of u. We will make use of one lemma, found in [4], to bound L^∞ norms of solutions in terms of L^∞ norms of source terms.

Lemma 4.1 ([4])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with C^2 boundary, and let $u : \overline{\Omega} \to \mathbb{R}^d$ be a $C^1(\overline{\Omega})$ flow with $\nabla\cdot u = 0$. Let ϕ solve the elliptic problem*

$$\begin{cases} -\triangle\phi + u\cdot\nabla\phi = g(x) & x\in\Omega \\ \phi = 0 & x\in\partial\Omega, \end{cases}$$

with $g \in L^p(\Omega)$, $p > d/2$. Then there exists a constant $C(\Omega, d, p) > 0$ depending on p and the domain Ω, but independent of the flow u, so that $\|\phi\|_{L^\infty(\Omega)} \le C\,\|g\|_{L^p(\Omega)}$.

We proceed to state the fast-flow convergence result precisely.

Theorem 4.2. *The solutions ϕ^{Au} to the problem (4.2) converge uniformly as $|A| \to \infty$ to the function $\overline{\phi^u} := f \circ \psi$, where f is given by*

$$f(h) := -\int_0^h \frac{\int_{\Omega_{\psi,m}} g\,dx}{\int_{\Omega_{\psi,m}} \triangle\psi\,dx}\,dm. \tag{4.3}$$

In particular, $\overline{\phi^u}$ is constant on level sets of ψ. Additionally, f is the unique solution of the Freidlin problem

$$\begin{cases} -\dfrac{1}{T(h)}\dfrac{d}{dh}\left(p(h)\dfrac{df}{dh}\right) = \overline{g}(h) \\ f(0) = 0, f(h) \text{ is bounded for } 0 \le h \le M \end{cases} \tag{4.4}$$

with the coefficients

$$T(h) := \int_{\partial\Omega_{\psi,h}} \frac{1}{|\nabla\psi|}\, d\sigma, \quad p(h) := \int_{\partial\Omega_{\psi,h}} |\nabla\psi|\, d\sigma,$$

and streamline-averaged source term

$$\overline{g}(h) := \frac{1}{\int_{\partial\Omega_{\psi,h}} \frac{1}{|\nabla\psi|}\, d\sigma} \int_{\partial\Omega_{\psi,h}} \frac{g}{|\nabla\psi|}\, d\sigma.$$

Remark 4.3. For the nonlinear problem $-\triangle\phi^{Au} + Au\cdot\nabla\phi^{Au} = \lambda g(\phi^{Au})$, assuming the stream function ψ of u admits a single nondegenerate maximimum, [4] showed that if $\lambda < \limsup_{A\to\infty} \lambda^*(Au)$ and the solutions ϕ^{Au} are taken to be minimal positive solutions, then there is a subsequence $A_n \to \infty$ with $\phi^{A_n u}$ converging strongly in $L^2(\Omega)$ to a $\overline{\phi^u}$ which satisfies a similar effective Freidlin problem.

Proof of theorem 4.2. We begin by changing coordinate systems. Choose (h, θ) coordinates where $h \in [0, M]$ specifies the level set of ψ and $\theta \in [0, 2\pi]$ gives a certain notion of angle around Ω. We define θ on $\partial\Omega$ to be normalized arclength, so that the total length is 2π, from an initial fixed point $p \in \partial\Omega$. To define θ on the interior of Ω, we consider the ordinary differential equation $\frac{d}{dt}X_t = \nabla h(X_t)$ with varied initial conditions $X_0 = x$ around $\partial\Omega$. Since there is a single nondegenerate maximum of ψ, we will have $\frac{d}{dt}h(X_t) > 0$ and $X_t \to x_0$ as $t \to \infty$, regardless of the initial position x. Then X_t follows characteristic lines inwards towards the maximum of ψ, and we set $\theta(X_t) := \theta(X_0)$. The value of $\theta(x_0)$ is inconsequential and left undefined. With these definitions we have $\nabla\theta \cdot \nabla h = 0$ and $|\nabla\theta|, |\nabla h| > 0$ except at x_0.

We consider trial guesses $\tilde{\phi}^{Au}$ of the form

$$\begin{cases} \tilde{\phi}^{Au} = \phi_0^u + \frac{1}{A}\phi_1^{Au} & x \in \Omega \\ \tilde{\phi}^{Au}(x) = 0 & x \in \partial\Omega \end{cases}$$

where $\phi_0^u = f(h)$ is constant along level sets of ψ and is independent of A, but ϕ_1^{Au} may depend on A. We insist that $f(0) = 0$ and hence that $\phi_1^{Au} = 0$ on $\partial\Omega$ as well. We will show eventually that f is given by (4.3). Applying the chain rule we have that

$$\nabla\phi_0^u = \nabla f(h) = f'(h)\nabla h$$

and

$$\begin{aligned} \triangle\phi_0^u =& \nabla \cdot (f'(h)\nabla h) \\ =& \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(f'(h)\frac{\partial h}{\partial x_i}\right) \\ =& \sum_{i=1}^{2} \left[f''(h)\left(\frac{\partial h}{\partial x_i}\right)^2 + f'(h)\frac{\partial^2 h}{\partial x_i^2}\right] \\ =& f''(h)|\nabla h|^2 + f'(h)\triangle h. \end{aligned}$$

Defining the differential operator $L^A := -\triangle + Au\cdot\nabla$ for each $A\neq 0$, we then have

$$\begin{aligned}
L^A\tilde{\phi}^{Au} &= -\triangle\tilde{\phi}^{Au} + Au\cdot\nabla\tilde{\phi}^{Au} \\
&= -\triangle(\phi_0^u + \frac{1}{A}\phi_1^{Au}) + Au\cdot\nabla(\phi_0^u + \frac{1}{A}\phi_1^{Au}) \\
&= -\triangle\phi_0^u + Au\cdot\nabla\phi_0^u - \frac{1}{A}\triangle\phi_1^{Au} + u\cdot\nabla\phi_1^{Au} \\
&= -\triangle\phi_0^u + Af'(h)u\cdot\nabla h - \frac{1}{A}\triangle\phi_1^{Au} + u\cdot\nabla\phi_1^{Au} \\
&= -\triangle\phi_0^u - \frac{1}{A}\phi_1^{Au} + u\cdot\nabla\phi_1^{Au}
\end{aligned}$$

where in the last equality we used that $u\cdot\nabla h = \nabla^\perp\psi\cdot\nabla\psi = 0$. We would like to choose ϕ_0^u, ϕ_1^{Au} so that

$$-\triangle\phi_0^u + u\cdot\nabla\phi_1^{Au} = g. \tag{4.5}$$

If $\tilde{\phi}^{Au}$ is chosen in this manner, then it will satisfy

$$L^A\tilde{\phi}^{Au} = g - \frac{1}{A}\phi_1^{Au}. \tag{4.6}$$

Since we constructed θ so that $\nabla\theta\cdot\nabla h = 0$, we have

$$u\cdot\nabla\theta = \nabla^\perp\psi\cdot\nabla\theta = \nabla^\perp h\cdot\nabla\theta = |\nabla h||\nabla\theta|.$$

It follows that

$$u\cdot\nabla\phi_1^{Au} = u\cdot\frac{\partial\phi_1^{Au}}{\partial\theta}\nabla\theta = (u\cdot\nabla\theta)\frac{\partial\phi_1^{Au}}{\partial\theta} = |\nabla h||\nabla\theta|\frac{\partial\phi_1^{Au}}{\partial\theta}$$

and so rearranging (4.5) tells us that we need

$$\frac{\partial\phi_1^{Au}}{\partial\theta} = \frac{g + \triangle\phi_0^u}{|\nabla h||\nabla\theta|}.$$

We then use the fact that our coordinate system is periodic in θ. Integrating around one period we have

$$\begin{aligned}
0 &= \int_0^{2\pi}\frac{g+\triangle\phi_0^u}{|\nabla h||\nabla\theta|}\,d\theta \\
&= \int_{\partial\Omega_{\psi,h}}\frac{g+\triangle\phi_0^u}{|\nabla h|}\,d\sigma \\
&= \int_{\partial\Omega_{\psi,h}}\frac{g + f''(h)|\nabla h|^2 + f'(h)\triangle h}{|\nabla h|}\,d\sigma \\
&= \int_{\partial\Omega_{\psi,h}}\frac{g}{|\nabla h|}\,d\sigma + f''(h)\int_{\partial\Omega_{\psi,h}}|\nabla h|\,d\sigma + f'(h)\int_{\partial\Omega_{\psi,h}}\frac{\triangle h}{|\nabla h|}\,d\sigma \\
&= \int_{\partial\Omega_{\psi,h}}\frac{g}{|\nabla h|}\,d\sigma + \frac{\partial}{\partial h}\left(f'(h)\int_{\partial\Omega_{\psi,h}}|\nabla h|\,d\sigma\right).
\end{aligned} \tag{4.7}$$

In the last step we differentiated the following equation

$$\int_h^M \int_{\partial\Omega\psi,t} \frac{\triangle h}{|\nabla h|}\, d\sigma\, dt = \int_{\Omega_{\psi,h}} \triangle h\, dx = \int_{\partial\Omega_{\psi,h}} \nabla h \cdot \hat{n}\, d\sigma = -\int_{\partial\Omega_{\psi,h}} |\nabla h|\, d\sigma$$

to obtain

$$\frac{\partial}{\partial h}\int_{\partial\Omega_{\psi,h}} |\nabla h|\, d\sigma = \int_{\partial\Omega,\psi,h} \frac{\triangle h}{|\nabla h|}\, d\sigma.$$

Recalling $|\nabla h| = |\nabla\psi| = |\nabla^{\perp}\psi| = |u|$, rearranging (4.7) and integrating in h from $m \in [0, M]$ to M then gives

$$0 - f'(m)\int_{\partial\Omega_{\psi,h}} |u|\, d\sigma = -\int_m^M \int_{\partial\Omega_{\psi,h}} \frac{g}{|\nabla h|}\, d\sigma\, dh = -\int_{\Omega_{\psi,m}} g\, dx$$

which tells us that

$$f'(h) = \frac{\int_{\Omega_{\psi,h}} g\, dx}{\int_{\partial\Omega_{\psi,h}} |u|\, d\sigma}.$$

This, together with the fact that

$$\int_{\partial\Omega_{\psi,h}} |u|\, d\sigma = \int_{\partial\Omega_{\psi,h}} |\nabla\psi|\, d\sigma = -\int_{\partial\Omega_{\psi,h}} \nabla\psi \cdot \hat{n}\, d\sigma = -\int_{\Omega_{\psi,h}} \triangle\psi\, dx$$

implies (4.3),

$$f(h) = -\int_0^h \frac{\int_{\Omega_{\psi,h'}} g\, dx}{\int_{\Omega_{\psi,h'}} \triangle\psi\, dx}\, dh'.$$

We take a moment to note that

$$-\int_{\Omega_{\psi,h}} \triangle\psi\, dx = \int_{\partial\Omega_{\psi,h}} |\nabla\psi|\, d\sigma = p(h)$$

and

$$-\frac{\partial}{\partial h}\int_{\Omega_{\psi,h}} g\, dx = -\frac{\partial}{\partial h}\int_h^M \int_{\partial\Omega_{\psi,h'}} \frac{g}{|\nabla h|}\, d\sigma\, dh' = \int_{\partial\Omega_{\psi,h}} \frac{g}{|\nabla h|}\, d\sigma = T(h)\bar{g}(h)$$

so that indeed f solves the Freidlin problem (4.4). We now use (4.3) to define ϕ_0^u. Using this initial guess we may then solve (4.5) for ϕ_1^{Au}, completing our construction of $\tilde{\phi}^{Au}$. Since $g + \triangle\phi_0^u$ is independent of A, we have that ϕ_1^{Au} may be chosen independently of A as well. With this observation it is then immediate that

$$\tilde{\phi}^{Au} = \phi_0^u + \frac{1}{A}\phi_1^{Au} \to \phi_0^u$$

uniformly as $|A| \to \infty$. Furthermore, comparing our guess $\tilde{\phi}^{Au}$ to the true solution ϕ^{Au}, we have by (4.6) that

$$L^A(\phi^{Au} - \tilde{\phi}^{Au}) = \frac{1}{A}\phi_1^{Au}.$$

As $\left\|\frac{1}{A}\phi_1^{Au}\right\|_\infty \to 0$, we may apply the lemma 4.1 in order to conclude that $||\phi^{Au} - \tilde{\phi}^{Au}||_\infty \to 0$. Hence $\phi^{Au} \to \phi_0^u$ uniformly as $|A| \to \infty$. □

5 Appendix

5.1 The stochastic representation theorem

We use what we refer to as the *stochastic representation theorem for the Dirichlet problem* generously throughout this paper. This section is dedicated to stating the specific version we are using and exactly what assumptions we require in order to invoke the representation theorem. The interested reader should turn to [21] for much more general statements as well as proofs of the theorems.

Let $\Omega \subseteq \mathbb{R}^d$ be nonempty, open, bounded, and connected. Let $u : \overline{\Omega} \to \mathbb{R}^d$ be Lipschitz. We do not require that $\nabla \cdot u = 0$. We know by Kirszbraun's theorem, cf. [24], [16], or [32], that there is a Lipschitz extension $u : \mathbb{R}^d \to \mathbb{R}^d$. We may also clip the resulting u to ensure that it is bounded. Thus we may suppose that u is defined on all of $\mathbb{R}^d$, globally Lipschitz, and bounded. Then, cf. theorems 5.2.5 and 5.2.9 in [21], for every $x \in \mathbb{R}^d$, strong existence and uniqueness holds for the SDE

$$\begin{cases} dX_t = -u(X_t)\,dt + \sqrt{2}\,dW_t, & \\ \qquad X_0 = x & P^x \text{ a.s.} \end{cases} \tag{5.1}$$

Because of strong existence, we may assume the underlying filtration is the augmented filtration generated by a single Brownian motion. The solution X is adapted, has continuous paths, and $E^x \|X_t\|^2_{\ell^2(\mathbb{R}^d)} < \infty$ for all $t \in [0, \infty)$. Under the assumptions we have made, the exit time τ_Ω of X from Ω satisfies $E^x \tau_\Omega < \infty$ for all $x \in \Omega$. Furthermore, the particular extension of u that we chose is irrelevant because we will only be evaluating $u(X_t)$ at times $t \le \tau_\Omega$, for which $X_t \in \overline{\Omega}$.

The Dirichlet problem on Ω with operator $L := \triangle - u \cdot \nabla$, boundary data $f \in C(\partial\Omega)$, and source term $g \in C(\overline{\Omega})$ is to find a $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{cases} -L\phi(x) = g(x), & x \in \Omega \\ \phi(x) = f(x) & x \in \partial\Omega. \end{cases} \tag{5.2}$$

We will typically need some or all of the following assumptions when solving (5.2) with stochastic methods:

- (SA1) $\Omega \subseteq \mathbb{R}^d$ is nonempty, open, bounded, and connected,
- (SA2) $u : \Omega \to \mathbb{R}^d$ is Lipschitz continuous,
- (SA3) $g \in C(\overline{\Omega})$,
- (SA4) $f \in C(\partial\Omega)$, and
- (SA5) Ω has the exterior sphere property, i.e. for every $x_0 \in \partial\Omega$ there is a ball B such that $\overline{B} \cap \Omega = \emptyset$ and $\overline{B} \cap \overline{\Omega} = \{x_0\}$. A sufficient condition for this is that Ω has C^2 boundary [14].

Now we are ready to state the representation theorem, theorem 5.7.2 in [21].

Theorem 5.1 (Stochastic representation)**.** *Under the assumptions (SA1)-(SA4), if we have that* $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *is a solution to the Dirichlet problem* (5.2)*, then* ϕ *admits the representation*

$$\phi(x) = E^x \left[f(X_{\tau_\Omega}) + \int_0^{\tau_\Omega} g(X_t)\, dt \right] \tag{5.3}$$

for all $x \in \overline{\Omega}$*. Moreover, if (SA5) holds, then* (5.3) *defines a* $C(\overline{\Omega}) \cap C^2(\Omega)$ *solution, and hence the unique solution, to the Dirichlet problem* (5.2)*.*

5.2 Maximum principles

In this section we show that some standard maximum principles from PDEs can be recovered as consequences of the stochastic representation theorem. Proving the strong maximum principle will require some intuitively obvious facts about Brownian motion. These results are quick consequences of the Cameron, Martin, Girsanov theorem, see section 3.5 in [21], so we state them first but give their proofs after the proofs of the maximum principles.

Theorem 5.2. *Let* W *be a standard* d*-dimensional Weiner process on the filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *satisfying the usual conditions, and let* $\gamma : [0, T] \to \mathbb{R}^d$ *be a Lipschitz continuous path starting from 0. Then for all* $\varepsilon > 0$*,* $P(\sup_{t \le T} |W_t - \gamma(t)| \le \varepsilon) > 0$*.*

Remark 5.3. Theorem 5.2 intuitively says that a Brownian motion will closely follow any given deterministic path with positive probability.

Corollary 5.4. *Let* W *be a standard* d*-dimensional Weiner process on the filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *satisfying the usual conditions. Let* D *be a bounded, open, connected region containing* 0*. Let* $S \subseteq \partial D$ *be such that there exists* $x_0 \in S$ *and* $r > 0$ *with* $B(x_0, r) \cap \partial D \subseteq S$*. Let* τ_D *denote the exit time of* W *from* D*, which is finite a.s. because* D *is bounded. Then* $P(W_{\tau_D} \in S) > 0$*.*

Corollary 5.5. *Let* W *be a standard* d*-dimensional Weiner process on the filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *satisfying the usual conditions. Let* D *be a bounded, open, connected region containing* 0*. Let* $B(x, r)$ *be a ball in* D *with positive distance from* ∂D*. Then for any* $s > 0$*, the probability that* W *will spend at least* s *time units inside* $B(x, r)$ *before exiting* D *is strictly positive.*

Theorem 5.6 (Weak maximum principle)**.** *Under the assumptions (SA1)-(SA2), if we have that* $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *and* $-\triangle\phi + u \cdot \nabla\phi \le 0$ *for all* $x \in \Omega$*, then* ϕ *achieves its maximum on* $\partial\Omega$*.*

Proof of theorem 5.6. First we handle the case $\phi \in C^2(\overline{\Omega})$. By the stochastic representation theorem 5.1 we have that ϕ admits the representation

$$\phi(x) = E^x \left[\phi(X_{\tau_\Omega}) + \int_0^{\tau_\Omega} [-\triangle\phi(X_t) + u(X_t) \cdot \nabla\phi(X_t)]\, dt \right]$$

for all $x \in \overline{\Omega}$, where τ_Ω is the exit time of X from Ω. Immediately we see that

$$\phi(x) \le E^x \phi(X_{\tau_\Omega}) \le \max_{s \in \partial\Omega} \phi(s)$$

where here we used $-\triangle\phi(X_t) + u(X_t)\cdot\nabla\phi(X_t) \le 0$ as well as the fact that $X_{\tau_\Omega} \in \partial\Omega$. Hence ϕ achieves its maximum on the boundary.

Now if $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$, take an exhausting sequence $\Omega_n \nearrow \Omega$ of open sets with $\overline{\Omega_n} \subseteq \Omega$. By the first part, the maximum of ϕ restricted to $\overline{\Omega_n}$ occurs on $\partial\Omega_n$, say $\phi|_{\overline{\Omega_n}}$ achieves this maximum at $x_n \in \partial\Omega_n$. By passing to a subsequence, we may assume $x_n \to y \in \overline{\Omega}$. By construction $y \in \partial\Omega$ and $\phi(y) = \max_{x\in\overline{\Omega}} \phi(x)$ by continuity of ϕ on $\overline{\Omega}$. □

Theorem 5.7 (Strong maximum principle)**.** *Under the assumptions (SA1)-(SA2), if we have that* $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$ *and* $-\triangle\phi + u\cdot\nabla\phi \le 0$ *for* $x \in \Omega$*, then* ϕ *attains its maximum at an interior point of* Ω *only if* ϕ *is constant.*

Proof of theorem 5.7. First we consider $\phi \in C^2(\overline{\Omega})$. Again we look at the stochastic representation provided by theorem 5.1,

$$\phi(x) = E^x\left[\phi(X_{\tau_\Omega}) + \int_0^{\tau_\Omega} [-\triangle\phi(X_t) + u(X_t)\cdot\nabla\phi(X_t)]\,dt\right].$$

Then since u is continuous and bounded, the Cameron, Martin, Girsanov theorem together with the Novikov condition tells us that, for each $x \in \mathbb{R}^d$, $\frac{1}{\sqrt{2}}X$ is a Brownian motion starting from x under a measure $\tilde{P}^x \ll P^x$. It is not important to us whether $P^x \ll \tilde{P}^x$ as we will be arguing $\tilde{P}^x(A) > 0$ in order to show $P^x(A) > 0$ for some specific sets A.

If ϕ is not constant on $\partial\Omega$, then we may choose a point $x_0 \in \partial\Omega$ such that $\phi(x_0) < \max_{x\in\overline{\Omega}} \phi(x)$. Using continuity of ϕ we may find a surface neighborhood $S := B(x_0, r) \cap \partial\Omega$ of x_0 such that $\phi(s) < \max_{x\in\overline{\Omega}} \phi(x)$ for all $s \in S$. Thus we have that $\tilde{P}^x(X_{\tau_\Omega} \in S) > 0$, and hence $P^x(X_{\tau_\Omega} \in S) > 0$, for all $x \in \Omega$ by corollary 5.4. In this case

$$\phi(x) \le E^x\phi(X_{\tau_\Omega}) < \max_{x\in\overline{\Omega}} \phi(x)$$

for all $x \in \Omega$, so ϕ does not achieve its maximum at any interior point.

Similarly, if we have $-\triangle\phi + u\cdot\nabla\phi < 0$ at some $x_0 \in \Omega$, then there is a ball $B(x_0, r)$ of positive distance from $\partial\Omega$ on which we have $-\triangle\phi + u\cdot\nabla\phi < 0$. Thus by corollary 5.5 we can show for $x \in \Omega$ that the $\tilde{P}^x$ probability, and hence the P^x probability, that X enters this ball and stays for 1 time unit before exiting Ω is strictly positive. Hence

$$\phi(x) < E^x\phi(X_{\tau_\Omega}) \le \max_{x\in\overline{\Omega}} \phi(x)$$

for all $x \in \Omega$, and again we find that ϕ does not attain its maximum at any interior point.

Otherwise, we know that ϕ is constant on $\partial\Omega$ and that $-\triangle\phi + u\cdot\nabla\phi = 0$ for all $x \in \Omega$, so

$$\phi(x) = E^x\phi(X_{\tau_\Omega}) = \max_{x\in\overline{\Omega}} \phi(x)$$

for all $x \in \overline{\Omega}$, and it follows that ϕ is constant.

Next we handle the case where $\phi \in C(\overline{\Omega}) \cap C^2(\Omega)$. Suppose ϕ achieves its maximum at $y \in \Omega$. Take an exhausting sequence $\Omega_n \nearrow \Omega$ of open sets with $\overline{\Omega_n} \subseteq \Omega$. For all n large, $y \in \Omega_n$, so by the first part that we proved $\phi|_{\overline{\Omega_n}}$ is constant for all n large. Since ϕ is continuous on $C(\overline{\Omega})$ and $\Omega_n \nearrow \Omega$, it follows that ϕ is constant on $\overline{\Omega}$. □

Now we give the proofs of the three auxiliary results: theorem 5.2, and corollaries 5.4 and 5.5.

Proof of theorem 5.2. Since γ is Lipschitz, γ is absolutely continuous with $\|\gamma'\|_{\ell^2(\mathbb{R}^d)} \in L^\infty([0,T])$. Since $\|\gamma'\|_{\ell^2(\mathbb{R}^d)} \in L^\infty([0,T])$ we have

$$E\left[\exp\left(\frac{1}{2}\int_0^T \|\gamma'(s)\|^2_{\ell^2(\mathbb{R}^d)}\, ds\right)\right] \leq \exp\left(\frac{1}{2}T\left\|\|\gamma'\|_{\ell^2(\mathbb{R}^d)}\right\|_{L^\infty([0,T])}\right) < \infty$$

so by the Cameron, Martin, Girsanov theorem we have that

$$\tilde{W}_t := W_t - \int_0^t \gamma'(s)\, ds = W_t - \gamma(t)$$

is a Brownian motion on $[0,T]$ under a probability measure $\tilde{P} : \mathscr{F}_T \to [0,1]$ which is equivalent to $P|_{\mathscr{F}_T}$. Let $\varepsilon > 0$ and assume for contradiction that $P(\sup_{t\leq T}|W_t - \gamma(t)| \leq \varepsilon) = 0$. Since $\tilde{P}$ is equivalent to P, we must have $\tilde{P}(\sup_{t\leq T}|\tilde{W}_t| \leq \varepsilon) = 0$. That is, the exit time $\tilde{\tau}$ of $\tilde{W}$ from $B(0,\varepsilon)$ is a.s. bounded by T. It follows by the optional sampling theorem that

$$0 = \tilde{E}[\tilde{W}_0] = \tilde{E}[\tilde{W}_{\tilde{\tau}}] = \varepsilon,$$

where $\tilde{E}$ denotes expectation under $\tilde{P}$. We have arrived at a contradiction, so it must be that actually $P(\sup_{t\leq T}|W_t - \gamma(t)| \leq \varepsilon) > 0$. □

Proof of corollary 5.4. The idea is to "leash" the Brownian motion, as in figure 5.2, and (with positive probability) lead it down a carefully selected path which will force it to hit S along the way. Choose $y \in D \cap B(x_0,r)$ and $z \in D^c \cap B(x_0,r)$. Use the path-connectedness of D and $B(x_0,r)$ to choose a smooth path $\gamma : [0,1] \to \mathbb{R}^d$ from 0 to y avoiding ∂D, then from y to z staying inside $B(x_0,r)$. We have that $\gamma([0,1])$ and $\partial D \setminus B(x_0,r)$ are disjoint compact sets, so their distance δ is strictly positive. Choose $\varepsilon := \min\{\delta/2, d(z,\partial D)/2\} > 0$. Then the ε-band around γ, $R := \left\{x \in \mathbb{R}^d : d(x,\gamma([0,1])) < \varepsilon\right\}$, only intersects ∂D at elements of S, and any path from 0 to z which stays in R must exit D at a point in S. We now apply theorem 5.2 to say that W will follow γ within ε distance, necessarily exiting D for the first time by hitting S, with strictly positive probability. □

Proof of corollary 5.5. We now understand how a leashing argument works, so the details will be suppressed. Choose a deterministic path γ avoiding ∂D that stays inside $B(x,r/2)$ for s time units and then exits D. By theorem 5.2, the Brownian motion will follow γ closely enough with positive probability. □

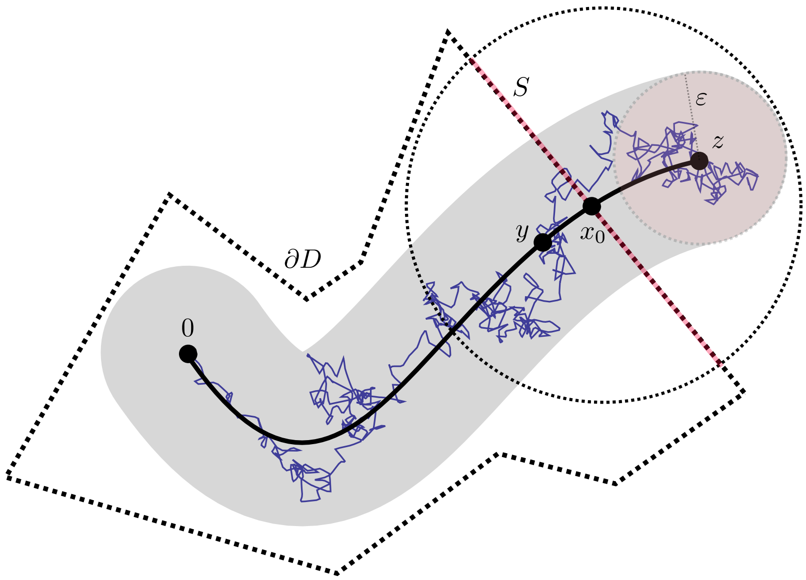

Figure 5.1: Leashing Brownian motion.

5.3 Liouville's theorem

Stochastic analysis can also be used to prove results from complex analysis. This proof is essentially taken from [29].

Theorem 5.8 (Liouville's theorem)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a bounded entire function. Then f is constant.*

Proof. Let $u(x, y) := \mathrm{Re} f(x + iy)$ be the real part of f. Since f is holomorphic, we know that $u \in C^2(\mathbb{R}^2)$ and $\triangle u = 0$. Let W be a two-dimensional Wiener process. By Itô's formula we have a.s. that

$$\begin{aligned} u(W_t) &= u(0) + \int_0^t \nabla u(W_s) \cdot dW_s + \frac{1}{2} \int_0^t \triangle u(W_s)\, ds \\ &= u(0) + \int_0^t \nabla u(W_s) \cdot dW_s. \end{aligned}$$

In particular, $u(W)$ is a bounded local martingale, and hence a true martingale. Since $u(W)$ is a continuous bounded martingale, the martingale convergence theorem tells us that $u(W_t)$ a.s. converges as $t \to \infty$. Suppose for contradiction that f is not constant. It must be that u is not constant, else the Cauchy-Riemann equations would imply f is constant. Hence by continuity we may choose disjoint nonempty balls B_1, B_2 such that $d(u(B_1), u(B_2)) > 0$. However, recurrence of the two-dimensional Wiener process implies that W will a.s. visit both of B_1, B_2 infinitely many times. This is a contradiction because it implies that a.s. $u(W_t)$ does not converge as $t \to \infty$. It follows that f must be constant. □

Corollary 5.9 (Fundamental theorem of algebra)**.** *Every nonconstant $p \in \mathbb{C}[z]$ has a zero.*

Proof. If $p \in \mathbb{C}[z]$ has no zeros then $1/p$ is a bounded entire function. □

5.4 Euler's product formula

We may also use probability to prove number-theoretic results. The following is part of an exercise from [35].

Theorem 5.10 (Euler's product formula)**.** *Define $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$. Then for all $s > 1$ we have*

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - p^{-s}\right)$$

where the product is taken over all primes $p \in \mathbb{N}$.

Proof. Fix $s > 1$ and choose a probability measure P and an $\mathbb{N}$-valued random variable X in such a way that $P(X = n) = n^{-s}/\zeta(s)$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ define E_n to be the event that n divides X. Then for $n \in \mathbb{N}$

$$P(E_n) = \sum_{k=1}^{\infty} P(X = kn) = \sum_{k=1}^{\infty} \frac{(kn)^{-s}}{\zeta(s)} = n^{-s}.$$

Next we claim that $\{E_p : p \text{ prime}\}$ is a mutually independent family. Indeed, for distinct primes $p_1, \ldots, p_k$ let $N := \prod_{i=1}^{k} p_i$ and we see

$$P\left(\bigcap_{i=1}^{k} E_{p_i}\right) = P\left(E_N\right) = N^{-s} = \prod_{i=1}^{k} p_i^{-s} = \prod_{i=1}^{k} P(E_{p_i}).$$

Then we are able to finish with

$$\frac{1}{\zeta(s)} = P(X = 1) = P\left(\bigcap_p E_p^c\right) = \prod_p (1 - p^{-s})$$

where p ranges over primes, and in the last equality we used the independence of the family $\{E_p : p \text{ prime}\}$. □

5.5 Viscosity solutions

We introduce viscosity solutions for second order partial differential equations. We found [8] to be helpful in learning about viscosity solutions, and much of the notation for viscosity solutions and ideas for proofs in this subsection stem from [8]. In this section $\Omega \subseteq \mathbb{R}^d$ is nonempty and open. It will be convenient to use the notation $D\phi$ for $\nabla\phi$ and $D^2\phi$ for the Hessian of ϕ. We also denote the set of real symmetric $d \times d$ matrices by $\mathbb{R}^{d\times d}_{\text{sym}}$.

Definition 5.11. Fix $F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}$ which is decreasing in the last component in the sense that if $M - N$ is positive semidefinite, then $F(x, r, p, M) \le F(x, r, p, N)$ for any valid x, r, p. We say that $\phi \in C(\Omega)$ is a *viscosity solution* of the equation $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ if both

(a) ϕ is a *viscosity supersolution* of $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$, i.e. $\phi \in C(\Omega)$ and for every $x_0 \in \Omega$ and every smooth test function $\varphi \in C^2(\Omega)$ such that $\phi - \varphi$ has a local minimum at x_0, we have $F(x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$, and

(b) ϕ is a *viscosity subsolution* of $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$, i.e. $\phi \in C(\Omega)$ and for every $x_0 \in \Omega$ and every smooth test function $\varphi \in C^2(\Omega)$ such that $\phi - \varphi$ has a local maximum at x_0, we have $F(x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0$.

Remark 5.12. It is often convenient to write that ϕ is a viscosity solution of $F(x, \phi, D\phi, D^2\phi) = 0$, or simply that ϕ is a viscosity solution for F, in order to save space.

Remark 5.13. For a viscosity solution ϕ, the notations $D\phi(x)$ and $D^2\phi(x)$ in the expression $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ are purely symbolic. We require that $\phi \in C(\Omega)$, but we do not require $D\phi$ or $D^2\phi$ to exist anywhere. In fact, we will see shortly that if we did require $\phi \in C^2(\Omega)$, the notion of viscosity solution would be equivalent to that of classical solution.

Remark 5.14. It is equivalent to require $\phi - \varphi$ to have a strict extremum at x_0 by adding or subtracting a $C^2(\Omega)$ function $h \geq 0$ for which $Dh(x_0) = 0$, $D^2h(x_0) = 0$ and $h(x) = 0$ if and only if $x = x_0$, e.g. $h(x) = |x - x_0|^4$. Then $D(\varphi \pm h) = D\varphi$ and $D^2(\varphi \pm h) = D^2\varphi$ at x_0, but the extremum will be strict. It is also equivalent to require $\varphi(x_0) = \phi(x_0)$ as $\varphi(x_0)$ does not appear anywhere in the definition.

Lemma 5.15. *If ϕ is a viscosity supersolution (subsolution) to both of the viscosity equations $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ and $G(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$, then ϕ is a viscosity supersolution (subsolution) to $(aF + bG)(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ for any $a, b \geq 0$.*

Proof. This follows immediately from the definition. □

Remark 5.16. Warning: knowing that ϕ is a viscosity solution to $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ does not necessarily imply that ϕ is a viscosity solution to $-F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$. We have built into the definition of viscosity solution the fact that F is decreasing in its last component in order to avoid as much ambiguity as possible. In statements like "ϕ is a viscosity solution of $-\triangle\phi = 0$," there is no ambiguity since we must be referring to $F(x, r, p, M) := -\operatorname{Tr}(M)$ and not $F(x, r, p, M) := \operatorname{Tr}(M)$ because the second choice is not decreasing in M. However, a statement like "ϕ is a viscosity solution of $|\phi'| - 1 = 0$" is ambiguous. In fact, [8] shows that if $|\phi'| - 1 = 0$ is interpreted with $F(x, r, p, M) := |p| - 1$ then $x \longmapsto |x| + 1$ is a viscosity solution on $(-1, 1)$, but if $|\phi'| - 1 = 0$ is interpreted with $F(x, r, p, M) := 1 - |p|$ then $x \longmapsto |x| + 1$ is not a viscosity solution.

The intuition behind the definition of viscosity solution comes from the fact that for $u, \varphi \in C^2(\Omega)$, if $\phi - \varphi$ has a local minimum at x_0 then $D(\phi - \varphi) = 0$ and $D^2(\phi - \varphi)$ is positive semidefinite at x_0, so our assumption on F implies $F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq F(x, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0))$. We formalize this intuition by showing that, under regularity assumptions, the notions of viscosity solution and classical solution are equivalent.

Lemma 5.17. *Suppose that $\phi \in C^2(\Omega)$. Then ϕ is a viscosity supersolution (subsolution) to $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ if and only if ϕ is a classical supersolution (subsolution) to $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$, i.e. for every $x \in \Omega$ we have $F(x, \phi(x), D\phi(x), D^2\phi(x)) \geq 0$ (≤ 0).*

Proof. We show the equivalence for supersolutions. ($\Longrightarrow$). Take $\varphi := \phi$ in the definition of supersolution. ($\Longleftarrow$). Let $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $\phi - \varphi$ has a local minimum at x_0 be given. We aim to show that $F(x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$. Since $\phi, \varphi \in C^2(\Omega)$ we have that $\phi - \varphi \in C^2(\Omega)$. Since Ω is open it follows that x_0 is an interior point of Ω, and hence $D(\phi - \varphi) = 0$ at x_0, i.e. $D\phi(x_0) = D\varphi(x_0)$. Similarly, since x_0 is a local maximum, we must have that $D^2(\phi - \varphi)$ is positive semidefinite at x_0. Using these two facts, the fact that F is decreasing in its last component, and the fact that ϕ is a classical supersolution, we have

$$\begin{aligned} 0 &\leq F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \\ &= F(x_0, \phi(x_0), D\varphi(x_0), D^2\phi(x_0)) \\ &\leq F(x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \end{aligned}$$

which finishes the proof. □

Thus we have shown the following equivalence between classical and viscosity solutions.

Theorem 5.18. *If $\phi \in C^2(\Omega)$, then ϕ is a classical solution to $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$ if and only if ϕ is a viscosity solution to $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$.* □

Now that we have established a link between classical and viscosity solutions, we present the main convergence result for viscosity solutions.

Theorem 5.19. *Suppose that, for each $n \in \mathbb{N}$, $F_n : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}_{sym} \to \mathbb{R}$ is decreasing in the last component. Further suppose that F_n is continuous for all $n \in \mathbb{N}$ and that $F_n \to F$ locally uniformly as $n \to \infty$. For each $n \in \mathbb{N}$ let ϕ_n be a viscosity supersolution (subsolution) of $F_n(x, \phi_n(x), D\phi_n(x), D^2\phi_n(x)) = 0$ such that $\phi_n \to \phi$ locally uniformly as $n \to \infty$. Then ϕ is a viscosity supersolution (subsolution) of $F(x, \phi(x), D\phi(x), D^2\phi(x)) = 0$. In particular, if the ϕ_n are viscosity solutions, then ϕ is a viscosity solution.*

Proof. We will show the result for supersolutions. Let $x_0 \in \Omega$ and let $\varphi \in C^2(\Omega)$ be given such that $\phi - \varphi$ has a local minimum at x_0. By remark 5.14 we may assume that it is a strict local minimum. We must show that $F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$. We built into the definition of viscosity supersolution that the ϕ_n must be continuous, and since $\phi_n \to \phi$ locally uniformly it follows that ϕ is continuous. Similarly, our local uniform convergence assumption for $F_n \to F$ implies F is continuous and decreasing in its last component. We claim that we can find a sequence (x_{n_k}) of points in Ω such that $x_{n_k} \to x_0$ and $\phi_{n_k} - \varphi$ has a local minimum at x_{n_k}. Choose $r > 0$ so that $\overline{B(x_0, r)} \subseteq \Omega$ and $\phi_n \to \phi$ uniformly on $\overline{B(x_0, r)}$. Then for each $n \in \mathbb{N}$, by continuity of $\phi_n - \varphi$ and compactness of $\overline{B(x_0, r)}$, choose an $x_n \in \overline{B(x_0, r)}$ such that $\phi_n - \varphi$ is minimized in $\overline{B(x_0, r)}$ at x_n. Again by compactness of $\overline{B(x_0, r)}$ we may extract a convergent subsequence $x_{n_k} \to y \in \overline{B(x_0, r)}$. We will show $y = x_0$. Since $x_0 \in \overline{B(x_0, r)}$, by definition of minimum we have that

$$\phi_{n_k}(x_{n_k}) - \varphi(x_{n_k}) \leq \phi_{n_k}(x_0) - \varphi(x_0).$$

By uniform convergence of $\phi_{n_k} \to \phi$ on $\overline{B(x_0, r)}$ combined with convergence of $x_{n_k} \to y$ we may take limits of both sides to obtain

$$\phi(y) - \varphi(y) \leq \phi(x_0) - \varphi(x_0)$$

which by the fact that x_0 is a strict local minimum implies that $y = x_0$. Now that we have a sequence of local minima approaching x_0, we may use the fact that the ϕ_{n_k} are viscosity supersolutions to say that

$$F_{n_k}(x_{k_n}, \phi_{n_k}(x_{n_k}), D\varphi(x_{n_k}), D^2\varphi(x_{n_k})) \geq 0.$$

Using that $\varphi \in C^2(\Omega)$ and again using that $x_{n_k} \to x_0$ and $\phi_{n_k} \to \phi$ uniformly on $\overline{B(x_0, r)}$ allows us to conclude that

$$(x_{k_n}, \phi_{n_k}(x_{n_k}), D\varphi(x_{n_k}), D^2\varphi(x_{n_k})) \to (x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)).$$

This fact, together with local uniform convergence of $F_n \to F$, implies

$$F(x_0, \phi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

which completes the proof. □

Viscosity solutions are just one type of weak solution to a PDE. Another important type of weak solution is *distributional solutions*. It is shown in [17] that, in fact, the notion of a viscosity solution is also equivalent to that of a distributional solution for continuous solutions. Ishii proves in [17] under mild smoothness and symmetry assumptions on coefficients of the uniformly elliptic L (which are satisfied for all L in this paper) that

Theorem 5.20 ([17])**.** *If $\phi \in C(\Omega)$, then $L\phi = f$ in the sense of distributions if and only if $L\phi = f$ in the viscosity sense.*

We will not go into detail about distributional solutions or the proof of theorem 5.20 in this paper.

6 References

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